The Kuga-Satake Construction: A Modular Interpretation

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Abstract

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Given a polarized complex K3 surface, one can attach to it a complex abelian variety, called Kuga-Satake variety. The Kuga-Satake variety is determined by the singular cohomology of the K3 surface; on the other hand, this singular cohomology can be recovered by means of the weight 1 Hodge structure associated to the Kuga-Satake variety. Despite the transcendental origin of this construction, Kuga-Satake varieties have interesting arithmetic properties. Kuga-Satake varieties of K3 surfaces defined over number fields descend to finite extension of the field of definition. This property suggests that the Kuga-Satake construction can be interpreted as a map between moduli spaces. More precisely, one can define a morphism, called Kuga-Satake map, between the moduli space of K3 surfaces and the moduli space of abelian varieties with polarization and level structure. This morphism, defined over a number field, is obtained by regarding the classical construction as a map between an orthogonal Shimura variety, closely related to the moduli space of K3 surfaces, and the Siegel modular variety. The most remarkable fact is that the Kuga-Satake map extends to positive characteristic for almost all primes, associating to K3 surfaces abelian varieties over finite fields. This can be proven applying a result by Faltings on the extension of abelian schemes and the good reduction property of Kuga-Satake varieties.

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To my sister,

Sofia

Contents

Chapter 1

Introduction

The classical Kuga-Satake construction arises in the transcendental setting. The goal is to associate to a polarized Hodge structure of K3 type a complex abelian variety. Given a polarized complex K3 surface X, the primitive part of the cohomology group $H^2(X,\mathbb{Z})$ is endowed with a Hodge structure of K3 type. Hence, we can attach to every K3 surface an abelian variety, called the Kuga-Satake variety. The relation between the K3 surface X and its Kuga-Satake variety is encoded in the isomorphism of algebras and Hodge structures

$$C^+(PH^2(X,\mathbb{Z})(1)) \simeq \mathrm{End}_C(H^1(\mathrm{KS}(X),\mathbb{Z})).$$

To phrase it in a different way, the primitive cohomology of a K3 surface can be described by means of Hodge structures associated to an abelian variety. Moreover, the Kuga-Satake variety is related to the geometry of the K3 surface. If X is a Kummer surface, that is a K3 surface arising from an abelian variety A, the Kuga-Satake variety of X is a power of the abelian variety A, up to isogeny.

In Chapter 3, we study the behaviour of the Kuga-Satake construction under deformation. We see that the constructions can be carried out for families of complex K3 surfaces. Given a K3 scheme $X \to S$, where S is a smooth, connected scheme over the complex numbers, we would like to construct an abelian scheme $A \to S$, corresponding, fiberwise, to the Kuga-Satake variety. The classical Kuga-Satake construction lets us define a Hodge variation of weight 1 over S, which is fiberwise polarizable. Up to replacing S with a finite étale covering, we can endow this Hodge variation with a global polarization. This allows us to construct an abelian scheme $A \to S$, which is the relative version of the Kuga-Satake construction.

From the Kuga-Satake construction in the relative setting, we deduce important arithmetic properties of the Kuga-Satake variety. For a K3 surface defined over a field of characteristic zero, the Kuga-Satake variety descends to a finite extension of this field. In addition, we can extend the construction to K3 surfaces in positive characteristic. Deligne proves in [Del82] that all K3 surfaces defined over a field of characteristic p admit a lifting X over a DVR S of mixed characteristic. We show that, applying the Kuga-Satake construction to the generic fiber of X, we obtain an abelian variety with potential good reduction. Thus, for a K3 surface X over a finite field k, we can find an abelian variety $\mathrm{KS}(X)$ over some extension of the basis, satisfying the relation

$$C^+(PH^2(X_{\bar{k}}, \mathbb{Z}_\ell)(1)) \simeq \mathrm{End}_C(H^1(\mathrm{KS}(X)_{\bar{k}}, \mathbb{Z}_\ell)),$$

for ℓ coprime to the characteristic of k. This isomorphism relates the ℓ-adic cohomology of a K3 surface to the Tate module of an abelian variety. This observation yields significant arithmetic consequences: it is the key step in the proof of the Weil conjecture for K3 surfaces presented by Deligne in [Del72].

In Chapter 5, I try to give a modular interpretation of the Kuga-Satake construction. Rather than a transcendental construction associating to a polarized K3 surface an abelian variety, we would like to define a Kuga-Satake map between moduli spaces of polarized K3 surfaces and the moduli space of abelian varieties. In order to achieve this goal, first, we construct the moduli space of K3 surfaces, following the classical approach for the moduli spaces of abelian varieties and curves of genus $g \geq 2$. The moduli space $\mathcal{P}_d$ of primitively polarized K3 surfaces of degree d is a separated Deligne-Mumford stack. For a K3 surface with a polarization of degree d, the primitive part of the ℓ-adic cohomology group, endowed with the cup-product, is isomorphic to the $\Lambda_d \otimes \mathbb{Z}_\ell$, where $\Lambda_d \subset \Lambda_0$ is a sub-lattice of the standard K3 lattice. Then we can introduce a notion of level structure for K3 surfaces as a marking of the second étale cohomology group, defined up to the action of a subgroup of finite index $\mathbb{K} \subset O(\Lambda_d)(\hat{\mathbb{Z}})$. Adding a level structure for an arithmetic group $\mathbb{K}$ to a polarized K3 surfaces, as in the case of abelian varieties, we obtain a separated algebraic space $\mathcal{P}_{d,\mathbb{K}}$.

In this setting, we can define a Kuga-Satake map, as a morphism of separated algebraic spaces over $\mathbb{C}$

$$\text{KS}\colon \mathcal{P}_{d,\mathbb{K}_N^{sp},\mathbb{C}} \to \mathcal{A}_{n,d',N,\mathbb{C}}$$

where $\mathcal{A}_{n,d',N}$ is the moduli space of polarized abelian varieties of genus $n = 2^{19}$, degree d' and level N. The construction of this map is realized via Shimura varieties. The transcendental construction associates to a polarized Hodge structure of K3 type with spin level N structure a polarized abelian variety with a level N structure. This can be interpreted as a map

$$ks\colon \mathbf{Sh}_{\mathbb{K}_N^{\mathrm{ad}}}(\mathrm{SO}(V), \Omega^{\pm}) \to \mathbf{Sh}_{\Lambda_N}(\mathrm{GSp}(L, \phi_a), \mathfrak{H}_n^{\pm})$$

for two $\mathbb{Q}$-vector spaces V and L, endowed with a symmetric and a symplectic form respectively. The orthogonal Shimura variety $\mathbf{Sh}_{\mathbb{K}_N^{\mathrm{ad}}}(\mathrm{SO}(V), \Omega^{\pm})$ parametrizes polarized Hodge structures of K3 type with a level structure, while the Shimura variety $\mathbf{Sh}_{\Lambda_N}(\mathrm{GSp}(L, \phi_a), \mathfrak{H}_n^{\pm})$ parametrizes polarized abelian varieties with a level structure, and can be identified with a component of the moduli space $\mathcal{A}_{n,d',N,\mathbb{C}}$. On the other hand, we can define a period map

$$j\colon \mathcal{P}_{d,\mathbb{K}_N^{sp},\mathbb{C}} \to \mathbf{Sh}_{\mathbb{K}_N^{\mathrm{ad}}}(\mathrm{SO}(V),\Omega^{\pm}),$$

associating to a polarized K3 surface with level structure its period. As Rizov proves in [Riz00], using an argument of complex multiplication for K3 surfaces, the period map j, a priori defined over $\mathbb{C}$, descends over $\mathbb{Q}$. The composition $\textsc{ks} = ks \circ j$ defines the so-called Kuga-Satake map over moduli spaces. The remarkable fact about this construction is that the map $\textsc{ks}$ descends to a finite extension E of $\mathbb{Q}$. Indeed, working with Shimura varieties lets us control the field of definition of the map ks rather easily, and, as we have seen, the period map j is defined over $\mathbb{Q}$.

The last part of Chapter 5 is devoted to showing that the morphism $\textsc{ks}$ can be extended in positive characteristic to an open part of $\mathrm{Spec}(\mathcal{O}_E)$, outside the primes dividing some $r > 0$. This extension argument is not formal. Let $\mathfrak{p} \subset \mathcal{O}_E[1/r]$ be a prime. Let U be a smooth covering of $\mathcal{P}_{d,\mathbb{K}_N^{\mathrm{sp}},\mathcal{O}_E[1/r]}$; it is a smooth scheme over $\mathcal{O}_E[1/r]$. The Kuga-Satake map in characteristic zero gives a morphism $U_E \to \mathcal{A}_{n,d,N,\mathcal{O}_E[1/r]}$. From the good reduction of Kuga-Satake varieties, one can easily deduce that this morphism extends to a subscheme of $V \subset U$ such that the complement has codimension greater than or equal to 2. Such a morphism corresponds to an abelian scheme $A_V \to V$ with polarization and level structure. The key step is applying a result of Faltings, in order to show that the abelian scheme A_V extends to all of U, and to define a morphism $U \to \mathcal{A}_{n,d,N,\mathcal{O}_E[1/r]}$. This lets us define an extension of the map $\textsc{ks}$ to a morphism

$$\textsc{ks}\colon \mathcal{P}_{d,\mathbb{K}_N^{sp},\mathcal{O}_E[1/r]} \to \mathcal{A}_{n,d,N,\mathcal{O}_E[1/r]},$$

that provides an extension of the Kuga-Satake map in positive characteristic.

Chapter 2

Generalities on K3 surfaces

In this chapter, we recall some facts about K3 surfaces, paying a particular attention to their cohomology groups. The second Betti cohomology group, for complex K3 surfaces, corresponds to the standard K3 lattice endowed with a Hodge structure, which classifies K3 surfaces up to isomorphism, as the Torelli Theorem states.

Most of the results presented can easily be deduced from Hodge decomposition for K3 surfaces over fields of characteristic zero. The extension of the results to positive characteristic requires a different approach, based on the lifting in characteristic zero, due to Deligne in [DI82].

2.1 Definition and first properties

Definition 2.1.1. A K3 surface over a field k is a smooth proper surface such that there is an isomorphism of sheaves

$$\mathcal{O}_X \simeq \omega_X, \qquad H^1(X, \mathcal{O}_X) = 0, \tag{2.1}$$

where $\omega_X = \Omega_X^2$ and $\mathcal{O}_X$ are the canonical and the structure sheaf respectively.

A smooth, proper surface is projective [Liu, Sect. 9.3.1, Rmk. 3.5]. Therefore, all algebraic K3 surfaces are automatically projective. K3 surfaces may be defined in the transcendental setting as well, as compact complex manifolds of dimension 2, satisfying the properties (2.1). Complex K3 surfaces are not necessarily projective. However, since projectivity plays a central role in our constructions, we will restrict ourselves to the case of algebraic K3 surfaces.

Example 2.1.2. Smooth quartics of $\mathbb{P}^3_k$ are K3 surfaces. Other examples of K3 surfaces are Kummer surfaces, arising from abelian varieties. Let A be an abelian variety and consider the involution map $i\colon A \to A$ sending $a \mapsto -a$. Take the blow-up of the abelian variety along the fixed points for the involution. The map i lifts to an involution $\tilde{\imath}$ over the blow-up; passing to the quotient modulo $\tilde{\imath}$, we obtain a K3 surface.

2.1.1 Line bundles on K3 surfaces

For K3 surfaces, the classical Riemann-Roch formula for a line bundle $\mathcal{L}$ on an algebraic surface can be reformulated as

$$\chi(\mathcal{L}) = \frac{1}{2}(\mathcal{L}, \mathcal{L}) + 2, \tag{2.2}$$

where $(\ ,\ )$ denotes the intersection pairing on the Picard group. In particular, given an ample line bundle $\mathcal{L}$ with self-intersection $(\mathcal{L}, \mathcal{L}) = 2d$, the Hilbert polynomial associated to the ample line bundle can be computed as

$$P_d(t) = \chi(\mathcal{L}^t) = dt^2 + 2. \tag{2.3}$$

Recall that the Néron-Severi group $\mathrm{NS}(X)$ is the quotient of the Picard group $\mathrm{Pic}(X)$ modulo numerically trivial line bundles. The intersection pairing on the Néron-Severi group $\mathrm{NS}(X)$ is even, non degenerate and of signature $(1, \rho - 1)$, where ρ is the rank of the Néron-Severi group; this follows from Hodge Index Theorem, as algebraic K3 surfaces are projective.

From the Riemann-Roch formula (2.2) we can deduce that numerically trivial line bundles on a K3 surface are trivial. Indeed, let $\mathcal{L}$ be a numerically trivial line bundle. By the Riemann-Roch formula, we have $\chi(\mathcal{L}) = 2$, which implies that either $h^0(\mathcal{L})$ or $h^2(\mathcal{L}) = h^0(\mathcal{L}^\vee)$ is non trivial. Then, either $\mathcal{L}$ is trivial or, up to replacing $\mathcal{L}$ with its dual, we can assume it is effective. Since $\mathcal{L}$ has positive intersection with any ample line bundle, $\mathcal{L}$ cannot be numerically trivial. Thus, the Picard group is isomorphic to the Néron-Severi group. It follows that the Picard group of a K3 surface is a finitely generated, free abelian group.

Proposition 2.1.3. *Let X be a K3 surface and $\mathcal{L}$ an ample line bundle on X. Then, $H^i(X, \mathcal{L}) = 0$ for $i > 0$. In addition, $\mathcal{L}^n$ is very ample for $n \geq 3$.*

Proof. The vanishing of the cohomology group $H^2(X, \mathcal{L})$ is an application of Serre duality

$$h^2(X, \mathcal{L}) = h^0(X, \mathcal{L}^\vee \otimes \omega) = h^0(X, \mathcal{L}^\vee) = 0,$$

where the last equality holds because $\mathcal{L}^\vee$ has trivial global sections, since its dual is ample. The rest of the statement is a classical result, for the proof of which we refer to [SD74]. □

2.1.2 Cohomology groups

We start considering the cohomology groups associated to a K3 surface in the complex setting. Let X be an algebraic K3 surface over $\mathbb{C}$. Regarding X as the complex manifold X^{an}, we can determine the singular cohomology of the K3 surface.

Proposition 2.1.4. *Let X be a complex K3 surface. The singular cohomology groups are torsion-free. The associated Betti numbers are*

$$b_0 = b_4 = 1, \qquad b_1 = b_3 = 0, \qquad b_2 = 22.$$

Proof. The isomorphism $H^0(X, \mathbb{Z}) \simeq H^4(X, \mathbb{Z}) \simeq \mathbb{Z}$ is a standard consequence of

Poincaré duality. The vanishing of the first cohomology group $H^1(X, \mathbb{Z})$ can be deduced from the long exact sequence induced by the exponential sequence of sheaves on X

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

and the fact that $H^1(X, \mathcal{O}_X) = 0$ by assumption. By Poincaré duality, this implies that the torsion-free part of $H^3(X, \mathbb{Z})$ vanishes. In order to determine the dimension of the second cohomology group, we can use Noether formula

$$2 = \chi(X, \mathcal{O}_X) = \frac{c_1^2 + c_2}{12}.$$

where $c_1^2 = (\omega, \omega) = 0$ and

$$c_2 = \sum_{i=0}^{4} b_i$$

corresponds to the Euler characteristic of X. From this, we can deduce that $c_2 = \sum_{i=0}^{4} b_i = 24$. Thus, $b_2 = 22$. Finally, we see that $H^2(X, \mathbb{Z})$ is torsion-free by considering the exact sequence

$$H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X),$$

induced by the exponential sequence, as both $H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$ and $H^2(X, \mathcal{O}_X)$ are torsion-free. This implies that $H^3(X, \mathbb{Z})$ is also torsion-free, because its torsion part can be identified with the torsion part of $H^2(X, \mathbb{Z})$. □

Therefore, the second cohomology $H^2(X, \mathbb{Z})$ is the only non-trivial Betti cohomology group for a K3 surface. The singular cohomology groups of a complex K3 surface admit a Hodge decomposition

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^q(\Omega_X^p)$$

(see Sect. 3.1), because K3 surfaces are Kähler complex manifolds. This result also holds for non algebraic complex K3 surfaces. The Hodge structure associated to the cohomology group $H^2(X, \mathbb{Z})$ is essential for carrying out the Kuga-Satake construction.

Proposition 2.1.5. *Let X be a complex K3 surface. The Hodge numbers associated to X are*

$$h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} = 0, \qquad h^{0,0} = h^{2,2} = h^{2,0} = h^{0,2} = 1, \qquad h^{1,1} = 20.$$

Proof. The property $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} = 0$ follows from the vanishing of $H^1(X, \mathcal{O}_X)$, the symmetry $h^{p,q} = h^{q,p}$ of the Hodge diamond and Serre duality. Similarly, $h^{0,0} = h^{2,2} = h^{1,1} = h^{0,2} = 1$ can be deduced from the fact that $h^0(X, \mathcal{O}_X) = 1$, and the assumption $\omega \simeq \mathcal{O}_X$. Finally, since we have the relation $b_2 = h^{2,0} + h^{0,2} + h^{1,1}$, we obtain $h^{1,1} = 20$. □

Remark 2.1.6. The previous result implies, in particular, that for complex K3 surfaces $H^0(X, \Omega^1_X) = 0$. From the relation $\mathcal{O}_X \simeq \Omega^2_X$, we deduce an isomorphism $\Omega^1_X \simeq \mathcal{T}_X$ between the sheaf of differentials and the tangent bundle. So we can restate the result saying that $H^0(X, \mathcal{T}_X) = 0$; in other words, vector fields over a K3 surface are trivial. This extends to every K3 surface over a field k of characteristic zero. In facts, it also holds in positive characteristic, but it is much harder to prove (see [SR76]).

The vanishing of $H^1(X, \mathcal{O}_X)$ implies, via the exponential exact sequence, that we can identify the Picard group $\mathrm{Pic}(X)$ with its image in $H^2(X, \mathbb{Z})$ via the first Chern class c_1. By Lefschetz Theorem the image of the Picard group corresponds to $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. In particular, we deduce that the rank of the Picard group is $\rho \leq 20$. In fact, the Picard number attains every value $0 \leq \rho \leq 20$ for some complex K3 surface.

The second cohomology group $H^2(X, \mathbb{Z})$ is naturally endowed with the cup-product, denoted by $(\ , \ )$, that makes $H^2(X, \mathbb{Z})$ into a unimodular lattice. The cup-product corresponds to the intersection pairing, if restricted to the Picard group. We have seen that the intersection pairing is even on the Picard group; the same holds for the cup-product on $H^2(X, \mathbb{Z})$, although this is not trivial to verify. We are interested in characterizing the unimodular lattice corresponding to $H^2(X, \mathbb{Z})$. Let Λ_0 be the K3 lattice, defined as

$$(\Lambda_0, \psi_0) = U^3 \oplus E_8(-1)^2,$$

where U is the standard hyperbolic lattice, and $E_8(-1)$ is the E_8 lattice with the form changed by a sign. By ψ_0 we denote the associated bilinear form.

Proposition 2.1.7. *Let X be a complex K3 surface. There is an isomorphism of quadratic lattices*

$$(H^2(X, \mathbb{Z}), (\ ,\ )) \simeq (\Lambda_0, \psi_0).$$

Proof. By the classification of even unimodular lattices, it is enough to prove that the cup-product defines an even pairing and that the signature is $(3, 19)$. For this, see [Huy]. □

Suppose now that the K3 surface X is endowed with an ample line bundle $\mathcal{L}$ of self-intersection $(\mathcal{L}, \mathcal{L}) = 2d$, for some $d > 0$. Assume that the line bundle $\mathcal{L}$ is not a power of any line bundle in the Picard group. Consider the orthogonal $PH^2(X, \mathbb{Z})$ to the class of $\mathcal{L}$ with respect to the cup product. The restriction of the form $(\ ,\ )$ to $PH^2(X, \mathbb{Z})$ induces a pairing, also denoted by $(\ ,\ )$, on the primitive part of the cohomology. Let e_1, f_1 be the standard basis of the first copy of U in the K3 lattice Λ_0 and define the sub-lattice (Λ_d, ψ_d) to be

$$(\Lambda_d, \psi_d) = \langle e_1 - df_1 \rangle \oplus U^2 \oplus E_8(-1)^2,$$

orthogonal to $e_1 + df_1$. The form restricted to this sub-lattice has signature (2, 19).

Theorem 2.1.8. *Let X be a complex K3 surface, endowed with a line bundle $\mathcal{L}$ with self-intersection $2d$ and no roots in the Picard group. There is an isomorphism of* *quadratic lattices*

$$(PH^2(X,\mathbb{Z}),(\ ,\ )) \simeq (\Lambda_d, \psi_d).$$

Proof. Let ℓ be the image of $c_1(\mathcal{L})$ under an isomorphism $H^2(X,\mathbb{Z}) \to \Lambda_0$. The lattice (Λ_0, ψ_0) has signature (3+, 19-); thus, we can apply the following result.

Proposition 2.1.9. *Let Λ be an even unimodular lattice of signature (n+, n-). Denote $r = \min\{n+, n-\}$. Let M be an even lattice, such that $\operatorname{rank}(M) < r$. There exists an embedding $M \hookrightarrow \Lambda$, unique up to automorphisms of the lattice Λ.*

This proves that there exists an automorphism of the lattice $g \in \mathrm{O}(\Lambda_0)$ such that $g(\ell) = e_1 + df_1$. In particular, $PH^2(X,\mathbb{Z}) \simeq \Lambda_d$. □

Let X be a K3 surface over a field k. We would like to characterize the étale cohomology groups of X. If k is a field of characteristic zero, we can deduce the conclusion from the complex case, while the positive characteristic case requires a different argument.

Proposition 2.1.10. *Let X be a K3 surface over k and $\ell \neq \mathrm{char(k)}$ be a prime. Then the cohomology groups $H^i(X_{\bar{k}}, \mathbb{Z}_\ell)$ are free $\mathbb{Z}_\ell$-modules of rank*

$$1,\ 0,\ 22,\ 0,\ 1.$$

Proof. If k is a field of characteristic zero, we can assume $k = \mathbb{C}$ and the result follows combining Prop. 2.1.4 with the comparison theorem between Betti cohomology and étale cohomology in [Del82]. If k is a field of positive characteristic and k is algebraically closed, we can apply the following theorem, proven by Deligne in [DI82, Cor. 1.8]:

Theorem 2.1.11. *Let X be a K3 surface over a finite field k, let W be the ring of Witt vectors of k, and let $\mathcal{L}$ be an ample line bundle on X. There is a DVR R of mixed characteristic, with $S = \operatorname{Spec}(R)$ finite over W, such that there exists a* *deformation of X in a proper smooth scheme $\mathcal{X} \to S$, and the line bundle $\mathcal{L}$ extends to an ample line bundle over S.*

We find a projective smooth scheme $\mathcal{X}$ over a DVR R of mixed characteristic and residue field k, such that the special fiber is isomorphic to X. Under these assumptions, K3 surfaces are stable under generization by Lemma 2.1.12. So if we let η be the generic point of $\mathrm{Spec}(R)$, we conclude, applying the proper smooth base change theorem for étale, cohomology, that

$$H^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}_\ell) \simeq H^i(X, \mathbb{Z}_\ell).$$

Then, since the result is proven for the generic fiber $\mathcal{X}_\eta$ in the characteristic zero case, it also holds for the special fiber X. □

Lemma 2.1.12. *Let $f\colon X \to S$ be a projective morphism of schemes of relative dimension 2. The condition of being K3 surfaces is open on the base.*

Proof. Suppose that there exists a point $s \in S$ such that the fiber X_s is a K3 surface. The set of points where the morphism is smooth determines an open subscheme U_1 of S. In particular, over the open subscheme U_1 the morphism f is flat, so the characteristic $\chi(\mathcal{O}_{X_u}) = 2$ and is constant for the geometric fibers $u \in U_1$. By the upper semi-continuity theorem, we have that the conditions

$$h^1(X_u, \mathcal{O}_{X_u}) = 0 \qquad\qquad h^0(X_u, \mathcal{O}_{X_u}) \leq 1$$

hold on an open subscheme U_2 of U_1. Therefore, by Serre duality, $h^0(X, \omega_u) = h^2(X, \mathcal{O}_{X_u}) \geq 1$ on the open subscheme U_2. If ω is the canonical sheaf, applying Serre duality again, we get

$$2 = \chi(\omega_u^\vee) = h^0(X_u, \omega_u^\vee) + h^1(X_u, \omega_u^\vee) + h^0(X_s, \omega_u^2)$$

and the conditions $h^0(X_u, \omega_u^2) \leq 1$ and $h^1(X_u, \omega_u^\vee) = 0$ hold on an open subscheme U of U_2. Hence, on U we have that $h^0(X_u, \omega_u)$ and $h^0(X_u, \omega_u^\vee)$. This implies that the canonical sheaf ω is trivial over U. Therefore, the open subscheme U corresponds to the locus of S where the fibers are K3 surfaces. □

As in the case of the singular cohomology, Poincaré duality for ℓ-adic cohomology provides a cup-product

$$(\ , \ )_\ell \colon H^2(X_{\bar{k}}, \mathbb{Z}_\ell) \times H^2(X_{\bar{k}}, \mathbb{Z}_\ell) \to \mathbb{Z}_\ell,$$

on the second cohomology of a K3 surface. The class of an ample line bundle $\mathcal{L}$ in $H^2(X_{\bar{k}}, \mathbb{Z}_\ell)$ defines by orthogonality a primitive part of the cohomology, with a quadratic form $(\ , \ )_\ell$. For complex K3 surfaces, the cup-product defined on the ℓ-adic cohomology coincides with the one on the singular cohomology , up to extension of the scalars to $\mathbb{Z}_\ell$. Hence, the ℓ-adic cohomology of a K3 surface over a field k of characteristic zero corresponds to

$$(H^2(X_{\bar{k}}, \mathbb{Z}_\ell), (\ , \ )_\ell) \simeq (\Lambda_0 \otimes \mathbb{Z}_\ell, \psi_{0,\ell}).$$

By the comparison theorems and arguing as in Prop. 2.1.10, we can see that the same result holds in positive characteristic. Similarly, given an ample line bundle $\mathcal{L}$ with self-intersection $2d$ and no roots in the Picard group, we conclude that

$$(PH^2(X_{\bar{k}}, \mathbb{Z}_\ell, (\ , \ )_\ell) \simeq (\Lambda_d \otimes \mathbb{Z}_\ell, \psi_{d,\ell}),$$

although the left hand side also has an action of the Galois group $\mathrm{Gal}(\bar{k}/k)$.

Chapter 3

The classical Kuga-Satake construction

In this chapter, we outline the classical Kuga-Satake construction. Given a polarized Hodge structure of K3 type, such as the second cohomology of a K3 surface, we attach to it a polarizable Hodge structure of weight 1. This gives rise to an abelian variety, called the Kuga-Satake variety. The relation between the Hodge structure of K3 type and its Kuga-Satake variety can be expressed by means of a Hodge cycle. Then we study some properties of the Kuga-Satake variety: how to determine polarizations and complex multiplication.

3.1 Generalities on Hodge structures

Let $A \subset \mathbb{R}$ be a subring of the real numbers and let V be a free A-module of finite type. We will consider the cases in which A is $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. We define an A-Hodge structure on V of weight $n \in \mathbb{N}$ to be a decomposition of the module $V_{\mathbb{C}} = V \otimes_A \mathbb{C}$ into complex subspaces of the kind

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

satisfying the equality $\overline{V^{p,q}} = V^{q,p}$. A Kähler complex manifold X, such as, for instance, a complex abelian varieties or a K3 surfaces, is endowed with an integral Hodge structure on the torsion-free part of its Betti cohomology groups $H^n(X,\mathbb{Z})$

$$H^n(X,\mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=n} H^q(X,\Omega^p),$$

where Ω^p is the sheaf of holomorphic differential p-forms.

Hodge structures can be regarded as representations of the Deligne torus. Let $\mathbb{S}$ be the real algebraic group obtained as the restriction of the multiplicative group over the complex numbers to the real numbers, that is

$$\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}.$$

More explicitly, the Deligne torus $\mathbb{S}$ satisfies the property $\mathbb{S}(A) = (A \otimes_{\mathbb{R}} \mathbb{C})^*$ for every $\mathbb{R}$-algebra A. By definition, the group $\mathbb{S}$ is an algebraic torus, whose real points correspond to $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$. The Deligne torus splits over $\mathbb{C}$ as $\mathbb{S}_{\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ (for references on algebraic groups, see [PR94]). Let $\mathbb{U}$ be the unit circle. If we let $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ be the morphism of algebraic corresponding to the inclusion $\mathbb{R}^* \subset \mathbb{C}^*$, there is an exact sequence

$$1 \to \mathbb{G}_{m,\mathbb{R}} \xrightarrow{w} \mathbb{S} \to \mathbb{U} \to 1.$$

The map $t\colon \mathbb{C}^* \to \mathbb{R}^*$, defined as $z \mapsto (z\bar{z})^{-1}$, is given by a morphism $t\colon \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$, such that there exists an exact sequences of real algebraic groups

$$1 \to \mathbb{U} \to \mathbb{S} \xrightarrow{t} \mathbb{G}_{m,\mathbb{R}} \to 1.$$

Let $h_{\mathbb{C}}\colon \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$ be a complex representation of the Deligne torus. It gives rise to a decomposition

$$V_{\mathbb{C}} = \bigoplus_{\chi \in X^*(\mathbb{S}_{\mathbb{C}})} V^{\chi}$$

where $X^*(\mathbb{S}_{\mathbb{C}}) = \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, \mathbb{G}_{m,\mathbb{C}})$ is the group of characters, which is a free abelian group of rank 2, because the Deligne torus splits over $\mathbb{C}$, generated by the characters $z, \bar{z}$. The summands V^{χ} are defined as the subspaces of $V_{\mathbb{C}}$ where the representation coincides with the one induced by the character χ. The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts both on the representation and on the group of characters. The action on $X^*(\mathbb{S}_{\mathbb{C}})$ exchanges the characters z and $\bar{z}$. The representation $h_{\mathbb{C}}$ is induced by a real representation $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ if the two actions of the Galois group are compatible. In other words, the representation is real if the decomposition

$$V_{\mathbb{C}} = \bigoplus_{(p,q) \in \mathbb{Z}^2} V^{p,q},$$

where $V^{p,q}$ is the subspace associated to the character $z^p \bar{z}^q$, satisfies the additional property $V^{p,q} = \overline{V^{q,p}}$. We say that $h\colon \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ is a real homogeneous representation of weight n if the composition $h \circ w : \mathbb{G}_m \to \mathrm{GL}(V_{\mathbb{R}})$ is given by $x \mapsto x^n$.

Proposition 3.1.1. *Let V be a free A-module, where $A \subset \mathbb{R}$ is a subring. Giving a Hodge structure of weight n on V is equivalent to giving a real homogeneous representation of weight n of the Deligne torus $h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$.*

Since Hodge structures correspond to representations of the Deligne torus, we can define direct sums, tensor products, duals, homomorphisms of Hodge structures via the corresponding operations in the category of representations. Given two Hodge structures (V, h) and (V', h'), we define a morphism of Hodge structures to be a morphism of A-modules $\phi\colon V \to V'$, such that $\phi_{\mathbb{R}}$ is equivariant for the action of $\mathbb{U}$ on $V_{\mathbb{R}}$ and $V'_{\mathbb{R}}$ induced by the representations h and h'. We say that ϕ is a strict morphism of Hodge structures if it is also $\mathbb{S}$-equivariant (then, in particular, the Hodge structures V and V' have the same weight). Strict morphisms of Hodge structures

(V,h) and (V',h') of weight n correspond to Hodge cycles for the natural Hodge structure of weight 0 on the A-module $\mathrm{Hom}(V,V')$:

$$\mathrm{Hom}^{\mathrm{Hg}}(V,V') = \mathrm{Hom}(V,V') \cap \mathrm{Hom}(V,V')^{0,0}.$$

The Tate twist $A(n) = (2\pi i)^n A$ is the A-Hodge structure associated to the representation given by $h(z) = (z\bar{z})^{-n}$. More generally, for a Hodge structure V, we define $V(n) = V \otimes A(n)$.

Definition 3.1.2. Let (V,h) be an A-Hodge structure of weight n. A polarization of weight n is a morphism of A-Hodge structures $\psi\colon V \otimes V \to A(-n)$, such that its real extension $\psi_{\mathbb{R}}$ defines a symmetric, positive definite form $\psi_{\mathbb{R}}(\ ,h(i)\ )$.

Direct computation shows that a polarization on a Hodge structure of weight n is $(-1)^n$-symmetric.

Example 3.1.3. Let X be a Kähler complex manifold of dimension d and let

$$[\omega] \in H^2(X,\mathbb{Q}) \cap H^{1,1}(X,\mathbb{R})$$

be a Kähler form. The Hodge-Riemann pairing

$$(\alpha,\beta) \mapsto (-1)^{n(n-1)^2} \int_X \alpha \wedge \beta \wedge \omega^{d-n}$$

provides a polarization on the primitive part $PH^n(X,\mathbb{Q})$ of the cohomology group $H^n(X,\mathbb{Q})$. In particular, for the middle cohomology $H^n(X,\mathbb{Q})$, the Hodge-Riemann pairing coincides with the restriction of the cup-product to the primitive part, hence it is defined topologically.

Example 3.1.4. Let $A = V/U$ be a complex abelian variety. We can recover its lattice taking the first homology group $H_1(A,\mathbb{Z}) = U$; the isomorphism of real vector spaces $U \otimes \mathbb{R} \cong V$ endows $U \otimes \mathbb{R}$ with a complex structure. This is equivalent to associating to U a Hodge structure of weight -1. Conversely, given U an integral Hodge structure of weight -1, we can attach to it the complex torus $U^{1,0}/U$. Moreover, polarizations of Hodge structures of weight -1 correspond to polarizations of abelian varieties and morphisms of complex tori can be identified with morphisms of their Hodge structures. Therefore, the functor $A \mapsto H_1(A, \mathbb{Z})$ gives an equivalence of categories between the category of abelian varieties and the category of polarizable weight -1-Hodge structures. The same equivalence of categories works replacing the first homology group with the first cohomology group, although the functor $A \to H^1(A, \mathbb{Z})$ is contravariant. It gives an equivalence between the category of polarizable Hodge structures of weight 1 and the category of abelian varieties.

3.2 The Kuga-Satake abelian variety

3.2.1 Clifford algebras

Let A be a commutative ring, V a free A-module and Q a quadratic form on V. We define the Clifford $C(V)$ algebra of (V, Q) to be a quotient of the tensor algebra $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ by the (two-sided) ideal

$$I(Q) = \langle v \otimes v - Q(v) \rangle_{v \in V}.$$

Since the ideal $I(Q)$ is generated by even terms with respect to the $\mathbb{Z}$-grading of $T(V)$, the Clifford algebra inherits a $\mathbb{Z}/2\mathbb{Z}$-grading. Hence, there is a decomposition

$$C(V) = C^+(V) \oplus C^-(V)$$

such that the even part $C^+(V)$ has a sub-algebra structure. The anti-involution ι of the tensor algebra defined by $v_1 v_2 \cdots v_r \mapsto v_r \cdots v_2 v_1$ preserves the ideal $I(Q)$, so it descends to $C(V)$ and $C^+(V)$. Assume now that $A = \mathbb{Q}$. Let the Clifford group of (V, Q) be

$$\mathrm{CSpin}(V) = \{x \in C^+(V)^* \mid xVx^{-1} \subset V\}$$

the group of invertible elements of the Clifford algebra fixing V under conjugation. The Clifford group is a connected algebraic group, defined over $\mathbb{Q}$. The Clifford group acts on V by conjugation, which gives rise to a morphism $\mathrm{CSpin}(V) \to \mathrm{GL}(V)$. The morphism factors through $\mathrm{SO}(V)$, because conjugation preserves the quadratic form

$$Q(xvx^{-1}) = (xvx^{-1}) \cdot (xvx^{-1}) = Q(v),$$

and the image is contained in $\mathrm{SO}(V)$ because $\mathrm{CSpin}(V)$ is connected. This gives rise to an exact sequence of algebraic groups

$$1 \to \mathbb{G}_m \xrightarrow{w} \mathrm{CSpin}(V) \xrightarrow{\mathrm{ad}} \mathrm{SO}(V) \to 1. \tag{3.1}$$

The anti-involution ι lets us define a norm on the Clifford algebra as $N(x) = x \cdot \iota(x)$. We let the Spin group be the subgroup of $\mathrm{CSpin}(V)$ of elements of trivial norm. Then we have an exact sequence

$$1 \to \mathrm{Spin}(V) \to \mathrm{CSpin}(V) \xrightarrow{t} \mathbb{G}_m \to 1,$$

where $t = N^{-1}$. The Spin group is, in fact, a simply connected algebraic group, which arises as a universal covering of $\mathrm{SO}(V)$. The Clifford group acts on the even Clifford algebra $C^+(V)$ in two ways: by left multiplication, defining the so-called spin representation $C^+(V)_{\mathrm{sp}}$, and by conjugation, giving the adjoint representation $C^+(V)_{\mathrm{ad}}$. In other words, we let the spin and adjoint actions on V be respectively, for all $x \in \mathrm{CSpin}(V)$ and $v \in C^+(V)$,

$$x *_{\mathrm{sp}} v = x \cdot v, \qquad\qquad x *_{\mathrm{ad}} v = x \cdot v \cdot x^{-1}.$$

The Clifford algebra endowed with the adjoint representation is isomorphic to the even exterior algebra $C^+(V)_{\mathrm{ad}} = \bigoplus_i \bigwedge^{2i} V$, where the right hand side comes with the action of $\mathrm{CSpin}(V)$ by conjugation.

Proposition 3.2.1. *There is a isomorphism of $\mathbb{Q}$-algebras and representations*

$$C^+(V)_{\mathrm{ad}} = \mathrm{End}_{C^+(V)^{\mathrm{op}}}(C^+(V)_{\mathrm{sp}}).$$

Proof. We can consider $C^+(V)$ as a $C^+(V)^{\mathrm{op}}$-right module (in fact the anti-involution ι let us identify the Clifford algebra with its opposite ring canonically). The map $L\colon v \mapsto L_v$ sending an element to the left multiplication map by it (which is indeed a morphism of $C^+(V)^{\mathrm{op}}$-modules), is an isomorphism of rings and it is also $\mathrm{CSpin}(V)$-equivariant:

$$L(x *_{\mathrm{ad}} v)(w) = L(xvx^{-1})(w) = (x *_{\mathrm{sp}} L_v)(w). \qquad \square$$

3.2.2 Spin representation and weight 1 Hodge structures

The Betti cohomology of a K3 surface with a polarization is a Hodge structure of K3 type. The goal of this section is to attach to a polarized Hodge structure of K3 type a Hodge structure of weight 1, to which we can associate an abelian variety. The trick is to lift a morphism of the Deligne torus in the special orthogonal group to the Clifford group and let the Clifford group act on the even Clifford algebra.

Definition 3.2.2. Let V be a $\mathbb{Q}$-vector space of dimension $n+2$, with an integral lattice $V_{\mathbb{Z}}$, endowed with a bilinear symmetric form B. Suppose that the quadratic form Q associated to B has signature $(n+, 2-)$. A morphism $h\colon S \to \mathrm{SO}(V_{\mathbb{R}})$ is a Hodge structure of K3 type if it induces a Hodge decomposition of weight 0 and type $\{(1,-1),(0,0),(-1,1)\}$ such that $h^{1,-1} = 1$ and Q is a polarization for V.

Note that this definition makes sense because if the Hodge structure V has weight 0 and Q defines a polarization, the representation h takes values in $\mathrm{SO}(V_{\mathbb{R}})$. Suppose we have a representation $h\colon \mathbb{S} \to \mathrm{SO}(V_{\mathbb{R}})$ making V into an integral Hodge structure of K3 type. We want to prove that the representation h can be lifted to a morphism into $\mathrm{CSpin}(V_{\mathbb{R}})$.

Proposition 3.2.3. *There exists a unique lifting* $\tilde{h}\colon \mathbb{S} \to \mathrm{CSpin}(V_{\mathbb{R}})$ *making the following diagram commutative*

$$\begin{array}{ccccc} \mathbb{G}_{m,\mathbb{R}} & \xrightarrow{w} & \mathbb{S} & \xrightarrow{t} & \mathbb{G}_{m,\mathbb{R}} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{G}_{m,\mathbb{R}} & \xrightarrow{w} & \mathrm{CSpin}(V_{\mathbb{R}}) & \xrightarrow{t} & \mathbb{G}_{m,\mathbb{R}} \end{array}$$

Proof. It is easy to see that, if the lifting exists, it is unique. To show the existence, consider the decomposition of $V_{\mathbb{R}} = V_0 \oplus V_1$, where

$$V_1 = (V^{1,-1} \oplus V^{-1,1}) \cap V_{\mathbb{R}}, \qquad V_0 = V^{0,0} \cap V_{\mathbb{R}}.$$

Let $u_1 + iu_2$ be a $\mathbb{C}$-basis of $V^{1,-1}$ with $u_1, u_2 \in V_1$. The subspace $V^{1,-1}$ is isotropic for the form Q and the restriction of Q to V_1 is negative definite, since Q is a polarization for the Hodge structure. This implies that, up to replacing u_1 and u_2 with scalar multiples, we have the relations

$$B(u_1, u_2) = B(u_2, u_1) = 0, \qquad B(u_1, u_1) = B(u_2, u_2) = -1.$$

Let J be the element $J := u_1 \cdot u_2$. Since u_1 and u_2 are orthogonal for the quadratic form,

$$J^2 = (u_1 u_2) \cdot (u_1 u_2) = -u_1 \cdot (u_2 \cdot u_2) \cdot u_1 = -Q(u_1)Q(u_2) = -1.$$

Let $z = a + bi \in \mathbb{C}$. We claim that the morphism

$$\mathbb{S}(\mathbb{R}) \to \mathrm{SO}(V)(\mathbb{R})$$
$$a + bi \mapsto \mathrm{ad}(a + bJ)$$

is well-defined, because $a + bJ$ belongs to $\mathrm{CSpin}(V_R)$ and it gives a lifting of h, by setting $\tilde{h}(a+bi) = a+bJ$. To show this, it suffices to compute $\mathrm{ad}(a+bJ)v$ for $v \in V$ and $a + bi \in \mathbb{U}$. Suppose first that $v \in V_0$. Then v and J are orthogonal, so we can compute

$$(a + bJ)v(a - bJ) = (a^2 + b^2)v = v$$

So the map $\tilde{h}$ preserves the subspace V_0 and the representation acts as the identity on it. Then, it suffices to check the definition of the map on $v = u_1, u_2$ in order to conclude. A direct computation shows that $u_1 \pm iu_2$ is an eigenvector of eigenvalue $(a \pm ib)^2$ for the conjugation by $a + bJ$. This let us conclude that $V_{\mathbb{R}}$ is closed under conjugation by $a + bJ$, which then belongs to $\mathrm{CSpin}(V_R)$ and $\tilde{h}$ is indeed a lifting of h. □

The element J is independent of the choice of $u_1, u_2 \in V_{\mathbb{R}}$. The construction that we carried out defines a lifting $\tilde{h} \colon \mathbb{S} \to \mathrm{CSpin}(V_{\mathbb{R}})$ and lets us introduce two Hodge structures on the even Clifford algebra $C^+(V)$. If we let the Deligne torus act on the real Clifford algebra via the spin representation, we get the spin Hodge structure $(C^+(V), h_{\mathrm{sp}})$. Via the adjoint representation, we recover the Hodge structure (V, h) as a sub-Hodge structure of $(C^+(V), h_{\mathrm{ad}})$. If V has an integral lattice $V_{\mathbb{Z}}$, the same argument works to define the corresponding integral Hodge structures

$$(C^+(V_{\mathbb{Z}}), h_{\mathrm{sp}}), \qquad\qquad (C^+(V_{\mathbb{Z}}), h_{\mathrm{ad}}).$$

Now, Proposition 3.2.1 can be rephrased by saying that there is an isomorphism of $\mathbb{Z}$-algebras and integral Hodge structures

$$(C^+(V_{\mathbb{Z}}), h_{\mathrm{ad}}) = \mathrm{End}_C((C^+(V_{\mathbb{Z}}), h_{\mathrm{sp}})), \tag{3.2}$$

where $C = C^+(V_{\mathbb{Z}})^{\mathrm{op}}$. From the isomorphism with the even exterior algebra, it is clear that the Hodge structure $(C^+(V_{\mathbb{Z}}), h_{\mathrm{ad}})$ is again of weight 0. On the other hand, the spin Hodge structure is defined by the complex structure determined by

$$J : C^+(V_{\mathbb{R}}) \to C^+(V_{\mathbb{R}})$$
$$v \mapsto J \cdot v$$

so it is a weight 1 Hodge structure of type $\{(1,0),(0,1)\}$.

3.2.3 Polarization and Kuga-Satake varieties

We have seen that, given an integral Hodge structure $(V_{\mathbb{Z}}, h,)$ of weight 0 and K3 type with respect to a symmetric bilinear form B, we can define a Hodge structure of weight 1 on the Clifford algebra $C^+(V_{\mathbb{Z}})$, that is a complex torus. The additional datum of a polarization gives rise to an abelian variety.

Proposition 3.2.4. *Let a be an element in $C^+(V_{\mathbb{Z}})$ such that $\iota(a) = -a$. Let $\phi_a\colon C^+(V_{\mathbb{Z}}) \times C^+(V_{\mathbb{Z}}) \to \mathbb{Z}$ be the morphism defined as*

$$(v, w) \mapsto \mathrm{Tr}(\iota(v) \cdot w \cdot a).$$

Then $\pm\phi_a$ is a polarization of the Hodge structure $(C^+(V_{\mathbb{Z}}), h_{\mathrm{sp}})$.

Proof. The result follows from an easy computation (cfr. [Gee00]). □

Notice that we can always find elements $a \in C^+(V)$ (then also in $C^+(V_{\mathbb{Z}})$ taking an integral multiple), such that $\iota(a) = -a$. It suffices to choose an orthogonal basis of a rational sub-space of dimension 2 where the quadratic form Q is negative definite, say e_1, e_2. If we set $a = e_1 \cdot e_2$, by orthogonality we have $\iota(a) = e_2 \cdot e_1 = -e_1 \cdot e_2 = -a$. However, the choice of this element is not unique, so the Hodge structure $(C^+(V_{\mathbb{Z}}), h_{\mathrm{sp}})$ is polarizable, but not canonically.

Theorem 3.2.5. *Given a polarized integral Hodge structure of weight 0 and K3 type $(V_{\mathbb{Z}}, h)$, with a bilinear symmetric form B, we can attach to it a polarizable weight*

1-Hodge structure such that the associated abelian varieties $\mathrm{KS}(V)$ *satisfies*

$$H^1(\mathrm{KS}(V), \mathbb{Z}) = (C^+(V_{\mathbb{Z}}), h_{\mathrm{sp}}).$$

This abelian variety is called the Kuga-Satake variety of the triple $(V_{\mathbb{Z}}, h, B)$*.*

From the Kuga-Satake abelian variety $\mathrm{KS}(V)$, we can recover the (rational) Hodge structure V, as a sub-Hodge structure of a tensor of $H^1(\mathrm{KS}(V), \mathbb{Q})$. Choose an element $v_0 \in V$, we can define an embedding $V \hookrightarrow \mathrm{End}(H^1(\mathrm{KS}(V), \mathbb{Q}))$ sending $v \mapsto L_v \circ R_{v_0}$. It is a morphism of Hodge structures, because it is equivariant for the action of $\mathrm{CSpin}(V)$. There is a natural identification of

$$\mathrm{End}(H^1(\mathrm{KS}(V), \mathbb{Q})) = H^1(\mathrm{KS}(V), \mathbb{Q})^\vee \otimes H^1(\mathrm{KS}(V), \mathbb{Q}).$$

Via the polarization we define an isomorphism between the right hand side and $H_1(\mathrm{KS}(V), \mathbb{Q})^{\otimes 2}(1)$, because it gives an isomorphism $H^1(\mathrm{KS}(V), \mathbb{Q})(1) \to H^1(\mathrm{KS}(V), \mathbb{Q})^\vee$. This yields an inclusion of Hodge structures

$$V \hookrightarrow H^1(\mathrm{KS}(V), \mathbb{Q})^{\otimes 2}(1).$$

Let us spell this out for the case of the cohomology of a K3 surface. Let X be a complex K3 surface, the cohomology $H^2(X, \mathbb{Z})(1)$ is a Hodge structure of weight 0. The choice of an ample line bundle $\mathcal{L}$ gives a class $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$, which defines by orthogonality the primitive cohomology $PH^2(X, \mathbb{Z})(1)$. The primitive cohomology $PH^2(X, \mathbb{Z})(1)$, with the bilinear form defined by the cup-product, is a Hodge structure of K3 type. Applying the previous construction to it, we find an abelian variety $\mathrm{KS}(X)$ and an embedding

$$PH^2(X, \mathbb{Q})(1) \hookrightarrow H^1(\mathrm{KS}(X), \mathbb{Q})^{\otimes 2}(1). \tag{3.3}$$

Notice that, by the Künneth formula, this is a sub-Hodge structure of $H^2(\mathrm{KS}(X) \times \mathrm{KS}(X), \mathbb{Q})$. In other words, *the primitive cohomology of the K3 surface is a sub-Hodge structure of the cohomology of an abelian variety.*

Abelian varieties arising as Kuga-Satake varieties satisfy a remarkable property. Since multiplication on the right by elements of $C^+(V)$ give endomorphisms of the weight 1 Hodge structure $(C^+(V), h_{\mathrm{sp}}) = H^1(\mathrm{KS}(V), \mathbb{Z})$, there is an inclusion

$$C^+(V)^{\mathrm{op}} \hookrightarrow \mathrm{End}(\mathrm{KS}(V)),$$

that is, the abelian variety $\mathrm{KS}(V)$ has complex multiplication by $C = C^+(V)^{\mathrm{op}}$. However, this does not imply that the Kuga-Satake variety has complex multiplication, in the sense that it is not necessarily simple and the Q-algebra of endomorphisms does not contain a CM field of the required degree.

Remark 3.2.6. Kuga-Satake varieties arising from K3 surfaces are, in general, far from being simple, as we can see by considering the Kuga-Satake variety associated to their transcendental lattice. Let X be a K3 surface with a polarization, define $V_{\mathbb{Z}} = PH^2(X, \mathbb{Z})(1)$ and let $N_{\mathbb{Z}} = H^{0,0} \cap PH^2(X, \mathbb{Z})(1)$ be the primitive part of the Néron-Severi sub-lattice of $PH^2(X, \mathbb{Z})(1)$. Consider T the orthogonal of N with respect to the polarization. Then we have an orthogonal decomposition of the Hodge structure

$$PH^2(X, \mathbb{Q})(1) = N \oplus T.$$

For the Clifford algebra, we have a decomposition

$$C^+(V) = C^+(N \oplus T) = C^+(N) \otimes C^+(T) \oplus C^-(N) \otimes C^-(T).$$

Since N is a Hodge structure of type $\{0, 0\}$, $N_{\mathbb{R}}$ is a trivial representation of the Deligne torus $\mathbb{S}$. So the morphism $h : \mathbb{S} \to \mathrm{SO}(V_{\mathbb{R}})$ factors through $\mathrm{SO}(T_{\mathbb{R}})$ and, similarly, the lifting $\tilde{h} : \mathbb{S} \to \mathrm{CSpin}(V_{\mathbb{R}})$ factors through $\mathrm{CSpin}(T_{\mathbb{R}})$. Therefore, $C^+(N_{\mathbb{R}})$

and $C^-(N_{\mathbb{R}})$ are trivial representation, and since $C^+(T)$ and $C^-(T)$ are isomorphic as Hodge structure via the multiplication on the right by an invertible element in V, we can conclude that

$$C^+(V) = C^+(T)^{2^{\rho-2}} \oplus C^-(T)^{2^{\rho-2}} \simeq C^+(T)^{2^{\rho-1}}$$

where ρ is the rank of the Néron-Severi group.

Remark 3.2.7 (Variants of the Kuga-Satake construction)*.* While we defined the Kuga-Satake construction for polarized Hodge structures of K3 type, we could have weakened this assumption. It suffices to have a $\mathbb{Z}$-Hodge structure (V, h) of weight 0 and type $\{(-1, 1), (0, 0), (1, -1)\}$ with $h^{1,-1} = 1$, endowed with a quadratic form Q, negative definite on V^1 and such that $Q(v) = 0$ for all $v \in V^{1,-1}$. For this reason, we can carry out the Kuga-Satake construction for the whole second cohomology group of a K3 surface $H^2(X, \mathbb{Z})(1)$, by means of the cup-product. However, considering polarized K3 surfaces will be more convenient for the Kuga-Satake construction over moduli spaces.

Another possibility is to perform the construction for the whole Clifford algebra, rather than restricting to the even Clifford algebra. The advantage in doing so would be to read the inclusion of Hodge structures in (3.3) as a consequence of (3.2).

Chapter 4

The Kuga-Satake construction in families

Our ultimate goal is to carry out the Kuga-Satake construction over moduli spaces of K3 surfaces and abelian varieties. In order to do this, we need to show that the Kuga-Satake construction can be defined for deformations of complex K3 surfaces, or, in other words, that the construction works for complex families. Furthermore, we will see that the relation 3.2 and the complex multiplication are preserved in families, up to passing to étale coverings. Using these results, we will deduce important arithmetic properties of Kuga-Satake varieties: the Kuga-Satake variety of a K3 surface defined over a field $K \subset \mathbb{C}$ descends to a finite extension of K. The Kuga-Satake variety of a K3 surface with good reduction has itself potential good reduction. The content of this chapter is essentially based on [Del72] and [And96].

4.1 Variation of Hodge structures and period domains

In this section we summarize the basic results in [Del69].

Definition 4.1.1. Let S be a complex manifold, $A \subset \mathbb{R}$ a subring. A Hodge variation is the datum of a local system of A-modules H, together with a holomorphic Hodge filtration $F^\bullet H$ of the vector bundle $H_{\mathcal{O}} = H \otimes_A \mathcal{O}_S$ satisfying the conditions:

i) the Gauss-Manin connection associated to the local system H verifies

$$\nabla F^p H_{\mathcal{O}} \subset \Omega^1_S \otimes \nabla F^{p-1} H_{\mathcal{O}};$$

ii) for every $s \in S$, the fibre $(H_s, F^\bullet H_s)$ defines a Hodge structure.

A polarization ψ of H is a locally constant morphism $\psi : H \otimes H \to \underline{A}$ of local systems on S such that, for all $s \in S$, the fibres ψ_s are polarizations of the Hodge structure H_s.

Now let $\pi : X \to S$ be a smooth, projective morphism of analytic spaces, with S smooth. The higher direct images $R^i\pi_*\mathbb{Z}$ are locally constant sheaves, and a projective embedding $X \hookrightarrow \mathbb{P}^n_S$ defines a global section

$$\eta \in R^2\pi_*\mathbb{Z},$$

corresponding to the first Chern class of the line bundle $\mathcal{O}(1)$, such that $\eta_s \in H^2(X_s, \mathbb{Z})$ induces a polarization on the primitive part of the cohomology. Hence, we obtain sheaves $P^i\pi_*\mathbb{Z}$, by taking the primitive part of $R^i\pi_*\mathbb{Z}$ with respect to η, with Hodge filtrations and a polarization induced by η itself.

Proposition 4.1.2. *The local systems $P^i\pi_*\mathbb{Z}$, with the Hodge filtration induced by Hodge decomposition, are variations of Hodge structures. Furthermore, the Hodge-Riemann pairing associated to η induces a polarization of the Hodge variation $P^i\pi_*\mathbb{Z}$.*

If the complex analytic spaces in the proposition arise as analytification of schemes $X \to S$ over $\mathbb{C}$, some of the objects we have described may be defined algebraically. Namely, the sheaves $R^i\pi_*\mathbb{C}$ and $P^i\pi_*\mathbb{C}$ with their Hodge filtration come from the hypercohomology of the algebraic de Rham complex, and the Gauss-Manin connection also admits an algebraic definition. On the other hand, the integral Hodge variations $R^i\pi_*\mathbb{Z}$ and $P^i\pi_*\mathbb{Z}$ have no algebraic counterpart.

Let $V_\mathbb{Z}$ be a free $\mathbb{Z}$-module of finite rank n, and let $V_\mathbb{R} = V \otimes \mathbb{R}$ be the associated real vector space. Let $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ be a function such that $d(p, q) = d(q, p)$, $d(p, q) = 0$ for almost all pairs (p, q) and $d(p, q) = 0$ if $p + q \neq n$. Fix a set of tensors T of $V_\mathbb{R}$, including an element t_0. We wish to parametrize the set X of Hodge structures h on $V_\mathbb{Z}$ satisfying

- h is a real Hodge structure for $V_\mathbb{R}$ with Hodge numbers $h^{p,q} = d(p, q)$
- the elements $t \in T$ are Hodge tensors for $V_\mathbb{R}$
- t_0 is a polarization for $V_\mathbb{R}$.

The set X can easily be identified with a subset of a product of grassmannians of suitable dimensions, from which it acquires a topology. Since we are parametrizing polarized Hodge structures, X can be endowed with a unique complex structure, for which each connected component is a hermitian symmetric domain. For this complex structure, the constant sheaf $\underline{V}_\mathbb{Z}$, with the Hodge structure induced by the points $h \in X$, is a Hodge variation (see [Del79]). We call X the period domain for the triple $(V_\mathbb{Z}, d, T)$.

4.2 Kuga-Satake construction for complex families

Let V be a $\mathbb{Q}$-vector space, with an integral lattice $V_\mathbb{Z}$, and a bilinear form B of signature $(n, 2)$. Let $\Omega^\pm$ be the period domain of Hodge structures of weight 0 and K3 type on $V_\mathbb{Z}$ such that B is a polarization. In this case, the period domain can easily be characterized as follows. Defining a Hodge structure of K3 type on $V_\mathbb{Z}$ amounts to giving a line $\langle\sigma\rangle$ for some $\sigma \in V_{\mathbb{C}}$, satisfying

$$B(\sigma, \sigma) = 0, \qquad\qquad B(\sigma, \bar{\sigma}) < 0.$$

Indeed, this determines a Hodge structure of K3 type, by setting $V^{1,-1} = \langle\sigma\rangle$, $V^{-1,1} = \langle\bar{\sigma}\rangle$ and defining $V^{0,0}$ as the orthogonal to $V^{1,-1} \oplus V^{-1,1}$ with respect to the quadratic form B. Equivalently, the choice of such a Hodge structure reduces to the choice of a 2-dimensional subspace of $V_{\mathbb{R}}$ where the form B is negative definite, with the additional datum of an orientation. Thus, the period domain $\Omega^{\pm}$ can be identified, as a differential manifold, with

$$\Omega^{\pm} \simeq \mathrm{O}(n, 2)/\mathrm{SO}(2) \times \mathrm{O}(n).$$

The period domain has, in fact, two connected components Ω^{+} and Ω^{-}, corresponding to the choice of an orientation on the negative plane, so we denote it by $\Omega^{\pm}$. We have seen that $\Omega^{\pm}$ comes with an integral and a rational Hodge variation given by the constant sheaf $\underline{V}_{\mathbb{Z}}$ and $\underline{V}$ respectively, with the obvious Hodge structure pointwise. Since every morphism $h : \mathbb{S} \to \mathrm{SO}(V_{\mathbb{R}})$ factors in a unique way through $\mathrm{CSpin}(V_{\mathbb{R}})$, the period domain $\Omega^{\pm}$ also parametrizes the set of morphisms $\tilde{h} \colon \mathbb{S} \to \mathrm{CSpin}(V_{\mathbb{R}})$, such that the composition

$$\mathrm{ad} \circ \tilde{h} \colon \mathbb{S} \to \mathrm{SO}(V)$$

is a Hodge structure of K3 type on $V_{\mathbb{Z}}$.

Let G be either $\mathrm{SO}(V)$ or $\mathrm{CSpin}(V)$. We denote $\mathrm{CSpin}(V)(\mathbb{Z}) = C^{+}(V_{\mathbb{Z}})^{*} \cap \mathrm{CSpin}(V)$.

Proposition 4.2.1. *Let $\Gamma \subset G(\mathbb{Z})$ be an arithmetic subgroup which acts freely on the period domain $\Omega^{\pm}$. Let L be a representation of G, with a sub-lattice $L_{\mathbb{Z}}$. Then the rational Hodge variation $H = \underline{L}$, together with the integral Hodge variation $H_{\mathbb{Z}} = \underline{L}_{\mathbb{Z}}$, defined on the period domain descends to Hodge variation H_{Γ} and $H_{\Gamma,\mathbb{Z}}$ on the quotient*

$\Omega^{\pm}/\Gamma$.

Proof. Denote by $\rho : G \to \mathrm{GL}(L)$ the (rational) representation. The constant local system $\underline{L}$ can be endowed with a Hodge structure on each fiber L_h by $\rho \circ h$, which induces a Hodge variation over $\underline{L}$. The action of Γ on $\Omega^{\pm}$ by conjugation, can be lifted to an action on the pair $(X, \underline{L})$ of complex space and local system with a Hodge variation as follows. For $\gamma \in \Gamma$, the action on the topological space is $\gamma * h = \gamma \cdot h \cdot \gamma^{-1}$, while the action on the locally constant system $\underline{L}$ is (fiberwise) defined as

$$(\gamma^* L)_h \to L_h$$
$$\gamma(l) = \rho(\gamma))l$$

This morphism is compatible with the Hodge structure, because, if l is an element of $(\gamma^* L)_h = L_{\gamma^{-1}h\gamma}$

$$h(z)(\gamma l) = \gamma \cdot (\gamma^{-1} h(z) \gamma l) = \gamma \cdot ((\gamma^{-1} h \gamma)(z) l).$$

The action of Γ on the period domain is free by assumption, hence it is free also on the local system. Therefore, the quotient of the period domain exists and the constant local system passes to a local system H_Γ on $\Omega^{\pm}$ with fiber L and monodromy factoring through Γ. So we have a diagram

$$\begin{array}{ccc} H & \xrightarrow{p_H} & H_\Gamma \\ \downarrow & & \downarrow \\ \Omega^{\pm} & \xrightarrow{p} & \Omega^{\pm}/\Gamma \end{array}$$

Since H is a Hodge variation and the action of Γ is induced by automorphisms of the Hodge variation, the local system H_Γ inherits the Hodge variation structure. Similarly, the integral Hodge variation $H_{\mathbb{Z}} = \underline{L}_{\mathbb{Z}}$ on the period domain $\Omega^{\pm}$ descends to the quotient $\Omega^{\pm}/\Gamma$. □

Let us make some remarks about the arithmetic subgroups of $\mathrm{CSpin}(V)$ and $\mathrm{SO}(V)$. The exact sequence in (3.1) yields the exact sequence

$$1 \to \{\pm 1\} \to \mathrm{CSpin}(V)(\mathbb{Z}) \xrightarrow{\mathrm{ad}} \mathrm{SO}(V)(\mathbb{Z}).$$

In the Clifford group, we can define a subgroup of the level N subgroup as

$$\Gamma_N^{\mathrm{sp}} = \{x \in \mathrm{CSpin}(V)(\mathbb{Z}) \mid x \equiv 1 \mod N\} \tag{4.1}$$

The previous exact sequence shows that, for $N \geq 3$, the morphism ad identifies Γ_N^{sp} with the image Γ_N^{ad}, which is a subgroup of finite index in the principal level N group

$$\Gamma_N = \{a \in \mathrm{SO}(V)(\mathbb{Z}) \mid a \equiv 1 \mod N\}. \tag{4.2}$$

Both Γ_N and Γ_N^{ad} are arithmetic groups, although Γ_N^{ad} might not be a congruence subgroup. By the Minkowski-Serre lemma, Γ_N^{ad} and Γ_N are torsion-free for $N \geq 3$. This ensures the existence of the quotient $\Omega^{\pm}/\Gamma_N^{\mathrm{sp}}$ and $\Omega^{\pm}/\Gamma_N$ as complex manifolds, which are, in fact, quasi-projective varieties by the Baily-Borel theorem, recalled in what follows.

Theorem 4.2.2 (Baily-Borel)**.** *Let X be a hermitian symmetric domain, Γ be a torsion-free arithmetic subgroup. Then the quotient X/Γ has a unique structure of quasi-projective variety. In addition, if S is a reduced scheme over $\mathbb{C}$ and $f : S^{\mathrm{an}} \to X/\Gamma$ a morphism of analytic spaces, then f is algebraic.*

Proof. See [BB66]. □

Let us apply these results to the case of the Clifford algebra associated to $(V_{\mathbb{Z}}, B)$, regarded as representation of the Clifford group. Denote by $L = C^+(V)$ the even Clifford algebra of the pair (V, B) and let $L_{\mathbb{Z}} = C^+(V_{\mathbb{Z}})$ be the integral lattice associated to the pair $(V_{\mathbb{Z}}, B)$. Consider L as the spin representation of $\mathrm{CSpin}(V)$.

The constant system $\underline{L}_{\mathbb{Z}}$ gives a Hodge variation on $\Omega^{\pm}$, corresponding to a family of abelian variety.

Proposition 4.2.3. *Let $\Omega^{\pm}$ be a connected component of the period domain associated to the pair (V, B). There exists a family of abelian varieties $a\colon A \to \Omega^{\pm}$ satisfying*

i) for each point $h \in \Omega^{\pm}$, the fiber A_h is the Kuga-Satake variety $\mathrm{KS}((V_{\mathbb{Z}}, h))$;

ii) the family A has complex multiplication by $C = C^{+}(V_{\mathbb{Z}})^{\mathrm{op}}$;

iii) we have an isomorphism of sheaves of algebras and Hodge variations $C^{+}(\underline{V}_{\mathbb{Z}}) \cong \mathrm{End}_C(R^1 a_* \mathbb{Z})$.

Proof. On the period domain $\Omega^{\pm}$ we can construct the Hodge variations $\underline{V}$ and $\underline{L}$, corresponding to the adjoint representation V and the spin representation L of $\mathrm{CSpin}(V)$ respectively. Similarly, we can define the integral Hodge variations $\underline{V}_{\mathbb{Z}}$ and $\underline{L}_{\mathbb{Z}}$, given by the sub-lattices $V_{\mathbb{Z}} \subset V$ and $L_{\mathbb{Z}} \subset L$. The integral Hodge variation $\underline{L}_{\mathbb{Z}}$ is a polarizable Hodge variation of weight 1, which gives rise to a family of abelian varieties such that $R^1 a_* \mathbb{Z} \simeq \underline{L}_{\mathbb{Z}}$. Multiplication on the right by elements in C gives endomorphism of the Hodge variation $\underline{L}_{\mathbb{Z}}$, so there is an embedding $C \to \mathrm{End}_{\Omega^{\pm}}(A)$. The isomorphism 3.2 works in families, since it is defined by constant local systems, so we can conclude

$$C^{+}(\underline{V}_{\mathbb{Z}}) \simeq \mathrm{End}_C(R^1 a_* \mathbb{Z}).$$

□

Proposition 4.2.4. *Let Γ be a torsion-free arithmetic subgroup of $\mathrm{CSpin}(V)(\mathbb{Z})$. Suppose that the Hodge variation associated to the family A admits a global polarization invariant under the action of Γ. Then the family of abelian varieties descends to an abelian scheme $a\colon A_{\Gamma} \to \Omega^{\pm}/\Gamma$ on the quotient, satisfying the previous properties.*

Proof. The constant sheaves $\underline{V}_{\mathbb{Z}}$ and $\underline{L}_{\mathbb{Z}}$ are induced by representation of $\mathrm{CSpin}(V)$, so by Proposition 4.2.1 they descend to the quotient $\Omega^{\pm}/\Gamma$, giving local systems $\underline{V}_{\Gamma,\mathbb{Z}}$ and $\underline{L}_{\Gamma,\mathbb{Z}}$. The sheaf $\underline{L}_{\Gamma}$ is a Hodge variation of weight 1 and it is polarizable, because, by assumption, there exists a polarization of $\underline{L}_{\mathbb{Z}}$ invariant under the action of Γ. In order to conclude, we need the following lemma.

Lemma 4.2.5. *Let S be a smooth scheme over $\mathbb{C}$. There is a one-to-one correspondence between abelian schemes $a\colon A \to S$ and polarizable integral Hodge variations of weight 1 on S.*

Proof. Given an abelian scheme $a\colon A \to S$ we can associate to it the Hodge variation $R^1 a_* \mathbb{Z}$, which is polarizable. Conversely, let $H_{\mathbb{Z}}$ be a polarizable integral Hodge variation of weight 1 over S and let $H_{\mathbb{Z},s}$ be the fiber of the local system over a point $s \in S(\mathbb{C})$. This gives rise to an analytic family of abelian varieties $a\colon A \to S$. By the Riemann existence theorem, we can take a finite étale covering S_N of S such that the monodromy of the local system H factors through Γ_N. This lets us define a holomorphic map $S_N \to \mathfrak{H}^{\pm}/\Gamma_N$, where $\mathfrak{H}$ is a Siegel space and it is, in fact, a morphism of schemes by the Baily-Borel theorem. The universal abelian variety $\mathcal{A} \to \mathfrak{H}^{\pm}/\Gamma_N$ gives, by pullback, an abelian scheme $A_N \to S_N$, which corresponds to the pullback of $A \to S$ via the étale covering $S_N \to S$. Thus, $A \to S$ is an abelian scheme. □

By the lemma, the Hodge variation $L_{\Gamma,\mathbb{Z}}$ defines an abelian scheme $A_\Gamma \to \Omega^{\pm}/\Gamma$. The complex multiplication by the $\mathbb{Z}$-algebra C and the isomorphism of sheaves descend to the quotient. □

Let S be a smooth, connected scheme over $\mathbb{C}$, with a polarized Hodge variation $(H_{\mathbb{Z}}, \psi)$ of K3 type, of fiber isomorphic to $(V_{\mathbb{Z}}, B)$. The datum of a polarization lets us construct the sheaf of the Clifford algebra $C^+(H_{\mathbb{Z}})$, and we would like to make the local system $C^+(H_{\mathbb{Z}})$ into a polarizable Hodge variation of weight 1. In order to do this, we need to define an action of $\mathrm{CSpin}(V)$ on the the local system $C^+(H)$, but this would require a coherent choice of a marking $\phi\colon V_{\mathbb{Z}} \to H_{\mathbb{Z},s}$, for all $s \in S$, which cannot be done unless the monodromy is trivial. The problem can be fixed replacing S with a finite étale covering.

Theorem 4.2.6. *Let S be a smooth, connected scheme over $\mathbb{C}$, and let $(H_{\mathbb{Z}}, \psi)$ be a polarized Hodge variation of K3 type. There exists a finite étale covering $\pi\colon S' \to S$ and an abelian scheme $a\colon A \to S'$ such that:*

i) for each point $s \in S'$, the fiber A_s corresponds to the Kuga-Satake variety $\mathrm{KS}(H_{\mathbb{Z},\pi(s)}, \psi)$;

ii) the family A has complex multiplication by a $\mathbb{Z}$-algebra C;

iii) we have an isomorphism of sheaves of algebras and Hodge variations

$$C^+(\pi^* H_{\mathbb{Z}}) \cong \mathrm{End}_C(R^1 a_* \mathbb{Z}).$$

Proof. Choose a marking $\phi\colon V_{\mathbb{Z}} \to H_{\mathbb{Z},s}$ of a fiber, for some $s \in S$; this choice induces a marking ϕ_t on all the fibers $t \in T$. By the Riemann existence theorem, there exists a finite étale covering $\pi\colon S_N \to S$ such that, for the inverse image $H_{N,\mathbb{Z}} = \pi^* H_{\mathbb{Z}}$ the monodromy $\pi_1(S_N, s_N) \to \mathrm{O}(V_{\mathbb{Z}})$ factors through Γ_N for $N \geq 3$. This let us define a holomorphic period map into the quotient of some connected component of the period domain

$$\begin{aligned} j_0\colon S &\to \Omega^{\pm}/\Gamma_N \\ t &\mapsto j_0(\phi_t^{-1}(H_{N,\mathbb{Z},t})) \end{aligned}$$

such that $H_{N,\mathbb{Z}} = j_0^*(V_{\Gamma_N,\mathbb{Z}})$, where $V_{\Gamma_N,\mathbb{Z}}$ is the sheaf descending from the universal Hodge variation $\underline{V_{\mathbb{Z}}}$ on $\Omega^{\pm}$. By the Baily-Borel theorem this map is, in fact, algebraic. Since $\Gamma_N^{\mathrm{sp}} \subset \Gamma_N$ is of finite order, we have that the projection morphism $\Omega^{\pm}/\Gamma_N^{\mathrm{sp}} \to \Omega^{\pm}/\Gamma_N$ is finite étale. Consider the fiber product S' in the diagram

$$\begin{array}{ccc} S' & \xrightarrow{j_0'} & \Omega^{\pm}/\Gamma_N^{\mathrm{sp}} \\ \downarrow{\scriptstyle \pi'} & & \downarrow \\ S_N & \xrightarrow{j_0} & \Omega^{\pm}/\Gamma_N \end{array}$$

On the scheme S' we have a local system $H'_{\mathbb{Z}}$ obtained as $H'_{\mathbb{Z}} = \pi'^*(H_{N,\mathbb{Z}}) = j_0'^*(V_{\Gamma_N^{\mathrm{sp}}}, \mathbb{Z})$. Given an element $a \in C^+(V_{\mathbb{Z}})$ such that $\iota(a) = -a$ the polarization of the period domain $\Omega^{\pm}$ defined as $\phi_a(v, w) = \pm\mathrm{tr}(\iota(v) \cdot w \cdot a)$ for $v, w \in L_{\mathbb{Z}}$ is invariant under the action of Γ_N^{sp}. Indeed, if $\gamma \in \Gamma_N^{\mathrm{sp}}$

$$\phi_a(\gamma * v, \gamma * w) = \pm\mathrm{tr}(\iota(\gamma v)\gamma w a) = \pm N(\gamma)\mathrm{tr}(\iota(v) w a) = N(\gamma)\phi_a(v, w) = \phi_a(v, w),$$

because $N(\gamma) = 1$ for all $\gamma \in \Gamma_N^{\mathrm{sp}}$. Thus, by Theorem 4.2.4, we have an abelian family $A_{\Gamma_N^{\mathrm{sp}}} \to \Omega^{\pm}/\Gamma_N^{\mathrm{sp}}$ satisfying the required relations. The pullback of $A_{\Gamma_N^{\mathrm{sp}}}$ via the map j_0^* defines an abelian scheme on S', with complex multiplication by $C = C^+(V_{\mathbb{Z}})^{\mathrm{op}}$ and the isomorphism of sheaves of $\mathbb{Z}$-algebras and Hodge variations as in the statement. □

4.3 Arithmetic of Kuga-Satake varieties

The Kuga-Satake construction has a transcendental origin. In this section, we are going to show that the Kuga-Satake variety of a K3 surface defined over $K \subset \mathbb{C}$ descends to a finite extension of K, and to study reduction properties of Kuga-Satake varieties. The argument involves the good behaviour of the Kuga-Satake construction under deformation and the so-called rigidity property of Clifford algebras.

Lemma 4.3.1. *Let k be a field, (V, B) a k-vector space with a bilinear form. Let $\Gamma \subset \mathrm{Spin}(V)$ be a Zariski-dense subgroup, and let a be an automorphism of $C^+(V)$ commuting with the action of Γ by conjugation. Then a is the identity.*

Proof. The automorphism a commutes with a Zariski-dense subset of $\mathrm{Spin}(V)$, so, without loss of generality, we can assume $\Gamma = \mathrm{Spin}(V)$. Moreover, we can suppose k is algebraically closed. Let $n = \dim(V)$. The adjoint representation $C^+(V)$ of $\mathrm{Spin}(V)$ can be identified with (see [Bou59]):

- $\mathrm{End}(W)$, for a simple $C^+(V)$-module, if n is odd;

- $\mathrm{End}(W_1) \times \mathrm{End}(W_2)$, for two non isomorphic $C^+(V)$-modules, if n is even.

In the second case, let $W = W_1 \oplus W_2$. Then we can see $C^+(V) \subset \mathrm{End}(W)$. By the Skolem-Noether theorem, there exists an automorphism $\bar{a}$ of W such that $a(x) = \bar{a} \cdot x \cdot \bar{a}^{-1}$ for every $x \in C^+(V)$. The fact that a commutes with the adjoint representation of $\mathrm{Spin}(V)$ boils down to asking that the commutator $(\bar{a}, \gamma)$ belongs to the center of $C^+(V)$. In both the odd and the even case we can conclude that $\det_W((\bar{a}, \gamma)) = 1$ and $\det_{W_i}((\bar{a}, \gamma)) = 1$ for $i = 1, 2$, respectively, which yields the conclusion that $(\bar{a}, \gamma) = 1$ by connectedness of $\mathrm{Spin}(V)$. Finally, this implies that $\bar{a}$ commutes with the representation W or W_i of Γ; but this representation is irreducible, so $\bar{a}$ has to be a homothety. Hence the automorphism a is trivial. □

Theorem 4.3.2 (Descent of the Kuga-Satake construction). *Let X_0 be a polarized K3 surface, defined over a field $K \subset \mathbb{C}$. Then, up to replacing K with a finite extension, there exists a scheme S of finite type over K, with a K3 scheme $f \colon X \to S$ and an abelian scheme $a \colon A \to S$ such that:*

i) there is a point $s \colon \mathrm{Spec}(K) \to S$ with $X_s \simeq X_0$;

ii) the abelian scheme A has complex multiplication by a $\mathbb{Z}$-algebra C;

iii) for every prime ℓ, there is a unique isomorphism of $\mathbb{Z}_\ell$-sheaves of algebras

$$C^+(P^2 f_* \mathbb{Z}_\ell(1)) \simeq \mathrm{End}_C(R^1 a_* \mathbb{Z}_\ell).$$

Proof. The descent of properties i) and ii) requires a standard argument, while for the third property we have to relate Betti cohomology and étale cohomology and to use the rigidity of automorphisms of the Clifford algebra. By [And96], there exists a smooth connected scheme T over K with a projective K3 scheme $f_T : X_T \to T$ and a point $t \colon \mathrm{Spec}(K) \to T$ with fiber $X_t \simeq X_0$, such that it satisfies the following assumption.

Assumption 4.3.3. Extending the scalars to $\mathbb{C}$ and considering the analytification of the morphism

$$f_{T,\mathbb{C}} : X_{T,\mathbb{C}} \to T_{\mathbb{C}}$$

the monodromy $\pi_1(T_{\mathbb{C}}, t_{\mathbb{C}}) \to \mathrm{O}(PH^2(X_{t,\mathbb{C}}, \mathbb{Z})(1), t_{\mathbb{C}})$ is of finite index.

Furthermore, up to replacing $T_{\mathbb{C}}$ with an étale covering, we can apply Theorem 4.2.6 and construct an abelian scheme $A_{T,\mathbb{C}}$ over $T_{\mathbb{C}}$, with complex multiplication $m_{T,\mathbb{C}}$ and the isomorphism of sheaves of $\mathbb{Z}$-algebras $\alpha_{T,\mathbb{C}}\colon C^+(P^2 f_{T,\mathbb{C}*}\mathbb{Z}(1)) \cong \mathrm{End}_C(R^1 a_{T,\mathbb{C}*}\mathbb{Z})$ over $T_{\mathbb{C}}$. By the comparison theorem between Betti and étale cohomology (see [Del82]), we can deduce an isomorphism of étale-sheaves of $\mathbb{Z}_\ell$-algebras, denoted by β_T,

$$\beta_{T,\mathbb{C}}\colon C^+(P^2 f_{T,\mathbb{C}*}\mathbb{Z}_\ell(1)) \simeq \mathrm{End}_C(R^1 a_{T,\mathbb{C}*}\mathbb{Z}_\ell).$$

As all the data (T, X_T, A_T, m_T) are defined over $\mathbb{C}$ by a finite number of equations, they are, in fact, defined on an extension L of K of finite transcendence degree. Let (U, X_U, A_U, m_U) be a model of the previous package of data over L. The morphism of sheaves β_T descends to an algebraic closure $\bar{L}$ of L. To check that it also descend over L, we need to prove that, given $u\colon \mathrm{Spec}(L) \to U$, the isomorphism

$$C^+(PH^2(X_{\bar{L}}, \mathbb{Z}_\ell)(1)) \simeq \mathrm{End}_C(H^1(A_{\bar{L}}, \mathbb{Z}_\ell)) \tag{4.3}$$

commutes with the monodromy action of $\pi_1^{\mathrm{alg}}(U, u)$ on both sides. By assumption, the monodromy $\pi_1(T_{\mathbb{C}}, t_{\mathbb{C}}) \to \mathrm{O}(PH^2(X_{t,\mathbb{C}}, \mathbb{Z})(1), t_{\mathbb{C}})$ is of finite index, so the image of the fundamental group contains a Zariski-dense subgroup Γ of the special orthogonal group. The isomorphism 4.3 commutes with the action of this Zariski-dense subgroup Γ, therefore it commutes with the action of the inverse image of Γ via the morphism $\mathrm{Spin}(V) \to \mathrm{SO}(V)$, which is a covering. So there is a Zariski-dense subgroup of $\mathrm{Spin}(V)$ commuting with the isomorphism 4.3. This implies, by Lemma 4.3.1, that the isomorphism is unique, in particular it commutes with the action of $\pi_1^{\mathrm{alg}}(U, u)$.

This shows that β_T descends to an isomorphism β_U of étale sheaves over U.

We can find a connected smooth scheme W over K, with fraction field L and such that the system (U, X_U, A_U, m_U) is the generic fiber of another system (S, X_S, A_S, m_S), where $S = T \times_K W$, $X_S = X_T \times_K W$ and the isomorphism of sheaves β_U extends to β_S over S. For a finite extension K' of K, the scheme W has a K'-rational point. Taking a finite extension K' of K, the scheme W admits a K'-rational point $w\colon \mathrm{Spec}(K') \to W$. The morphism $(t'_K, w)\colon \mathrm{Spec}(K') \to S$ defines a point over K' with fiber $X_{0,K'}$.

□

The previous theorem yields important arithmetic consequences. Given a K3 surface over a field K, it permits finding a model for its Kuga-Satake variety over a finite extension of K. Moreover, if we assume that the K3 surface has good reduction over the spectrum of a henselian DVR, then the same holds for its Kuga-Satake variety.

Theorem 4.3.4. *Let X be a K3 surface defined a field $K \subset \mathbb{C}$. Then the abelian variety of $X_{\mathbb{C}}$ has a model over a finite extension of K.*

Proof. Consider the morphism $\varphi\colon W_{\mathbb{C}} \to S$, for $s = (t_{\mathbb{C}}, \mathrm{id}_W)$. One argues as in [And96, Prop. 5.5] that the fibers of the morphism

$$A_S \times_S W_{\mathbb{C}} \to W_{\mathbb{C}}$$

are isomorphic complex abelian varieties, by means of the isomorphism of sheaves $\varphi^*\beta_S$. Since the generic fiber is isomorphic to the Kuga-Satake variety of X_0, the choice of a K'-rational point in W provides a model for the Kuga-Satake variety.

□

Theorem 4.3.5. *Let $S = \mathrm{Spec}(R)$, where R is a henselian DVR of fraction field K and residue field k, and let $X \to S$ be a polarized K3 scheme. The Kuga-Satake variety*

$A_{K'}$ of the generic fiber X_K, defined on a finite extension K' of K, has potentially good reduction.

Proof. By Theorem 4.3.2, the Kuga-Satake variety is defined over a finite extension K' and there is an isomorphism of $\mathrm{Gal}(K'/K)$-modules

$$C^+(PH^2(X_{\bar{K}}, \mathbb{Z}_\ell)(1)) \simeq \mathrm{End}_C(H^1(A_{\bar{K}}, \mathbb{Z}_\ell)), \tag{4.4}$$

for any prime $\ell \neq \mathrm{char}(k)$. By proper smooth base change theorem, as the morphism $X \to S$ is proper and smooth, there is an isomorphism

$$H^i(X_{\bar{K}}, \mathbb{Z}_\ell)(1) = H^i(X_{\bar{k}}, \mathbb{Z}_\ell)(1)$$

invariant for the action of $\mathrm{Gal}(\bar{K}/K)$ on the left and $\mathrm{Gal}(\bar{k}/k)$ on the right. This implies that the action of the inertia group I on $C^+(PH^2(X_{\bar{K}}, \mathbb{Z}_\ell)(1))$ is trivial, so the same holds for the inertia action on $\mathrm{End}_C(H^1(A_{\bar{K}}, \mathbb{Z}_\ell))$. The action of the inertia group on $H^1(A_{\bar{K}}, \mathbb{Z}_\ell)$ commutes with the complex multiplication by C, so it factors through $\mathrm{End}_C(H^1(A_{\bar{K}}, \mathbb{Z}_\ell))$, isomorphic to $C^+(V_{\mathbb{Z}_\ell})$ where $V_{\mathbb{Z}_\ell} = PH^2(X_{\bar{K}}, \mathbb{Z}_\ell)$. The fact that the representation $\mathrm{End}_C(H^1(A_{\bar{K}}, \mathbb{Z}_\ell))$ is unramified implies that the action of $\mathrm{Gal}(\bar{K}/K)$ factors through the center of $C^+(V_{\mathbb{Z}_\ell})$. On the other hand, by the local monodromy theorem, there is a subgroup of finite index in I such that it acts unipotently on $H^1(A_{\bar{K}}, \mathbb{Q}_\ell)$. Provided that we take a finite extension K'' of K', we can assume the inertia action unipotent, hence trivial. Therefore, by Néron-Ogg-Shafarevich criterion, for which we refer to [TS68, Thm.1, page 493], the Kuga-Satake variety $A_{K''}$ has good reduction. □

Chapter 5

Moduli spaces of K3 surfaces

Since polarized abelian varieties have non-trivial automorphisms, their moduli functor cannot be representable by a scheme or an algebraic space. Thus, we have to consider the representability in a different setting, taking automorphisms into account. We define the category fibered in groupoids $\mathcal{A}_{g,d}$, whose objects are abelian varieties of genus g and with a polarization of degree d. A classical result tells us that this category fibered in groupoids is a Deligne-Mumford stack.

In this chapter, we use similar techniques to prove that the category fibered in groupoids of polarized K3 surfaces is also representable as a Deligne-Mumford stack. In addition, we define the notion of level structures for polarized K3 surfaces. For abelian varieties, a level N stucture is the choice of an isomorphism between the group of N-torsion points and the standard symplectic module over $\mathbb{Z}/N\mathbb{Z}$. We extend this notion to K3 surfaces, defining a level structure as a marking of the primitive cohomology group, up to the action of a level N group. Finally, we show that polarized K3 surfaces with a sufficiently fine level structures have no non-trivial automorphisms. Thus, the moduli functor of polarized K3 surfaces with level structures is representable by an algebraic space.

5.1 Preliminaries

We recall some definitions and results which will be important in the construction of the moduli stack of polarized K3 surfaces. The proofs rely mainly on classical arguments, adapted to the case of K3 surfaces.

5.1.1 Picard schemes and automorphisms

Definition 5.1.1. A proper and smooth morphism of schemes $X \to S$ whose geometric fibers are K3 surfaces is called a K3 scheme. A K3 space is an algebraic space X over a scheme S with an étale covering $S' \to S$ such that $X' \times S' \to S'$ is a K3 scheme.

Let $\pi\colon X \to S$ be a K3 space. Consider the relative Picard functor

$$\mathrm{Pic}_{X/S}\colon \mathbf{Sch} \to \mathbf{Group}$$

given by the fppf-sheafification of the functor associating to a scheme T the group $\mathrm{Pic}_T(X_T)$ of isomorphism classes of line bundles over $X \times_S T$, modulo pullbacks of line bundles over T.

Proposition 5.1.2. *Let $\pi\colon X \to S$ be a K3 space.*

i) The relative Picard functor $\mathrm{Pic}_{X/S}$ is representable by a separated algebraic space, locally of finite presentation over S.

ii) Suppose that $\pi\colon X \to S$ is a K3 scheme and there is a vector bundle $\mathcal{E}$ over S with a closed immersion $X \hookrightarrow \mathbb{P}(\mathcal{E})$. Then the relative Picard functor is representable. Moreover, for every $n > 0$, the morphism $[n] : \mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S}$ given by $\mathcal{L} \mapsto \mathcal{L}^n$ is a closed immersion.

Proof. See [Riz00, Section 1.3]. □

The condition in part ii) is automatically satisfied for a K3 scheme $X \to S$ up to passing to an étale covering, provided that the scheme $X \to S$ is polarized.

Definition 5.1.3. Let X be a K3 surface over a field k. A polarization is a line bundle L over X for which $L_{\bar{k}}$ is ample. A polarization is primitive if the line bundle L does not have any roots in $\mathrm{Pic}_{\bar{k}}(X_{\bar{k}})$. A (primitive) polarization for a K3 space $X \to S$ is a section λ of $\mathrm{Pic}_{X/S}(S)$ such that for all geometric points $s \to S$ the fiber λ_s is a (primitive) polarization for X_s.

Remark 5.1.4. Let $\pi\colon X \to S$ be a K3 scheme with a polarization λ. One can show that, for K3 schemes, the relative Picard functor is a sheafification with respect to the étale topology. Then, in particular, there exists an étale covering $S' \to S$ such that the $\lambda_{S'} = \mathcal{L}$ is the class of a line bundle on $X \times_S S'$. Since for every geometric point $s \in S$ the cohomology $H^1(X_s, \mathcal{L}_s)$ vanishes, the line bundle $\pi_*\mathcal{L}$ is locally free. Moreover, as all fibers of $\mathcal{L}^3$ are very ample, there is a closed immersion $X_s \hookrightarrow \mathbb{P}(H^0(X_s, \mathcal{L}_s))$, which implies that the sheaf $\mathcal{L}^3$ is relatively very ample. Hence, there is a closed immersion $X \hookrightarrow \mathbb{P}(\pi_*\mathcal{L}^3)$.

Suppose that $\pi\colon X \to S$ is a K3 space. We can also define the functor

$$\mathrm{Aut}_{X/S}\colon \mathbf{Sch}/S \to \mathbf{Group}$$

associating to a scheme T over S the group $\mathrm{Aut}_T(X_T)$. This functor plays a central role in the representability of the moduli space of abelian varieties as a Deligne-Mumford stack.

Proposition 5.1.5. *The functor* $\mathrm{Aut}_S(X)$ *is representable by a separated group scheme locally of finite type over* S*.*

Proof. Without loss of generality we can assume that $\pi\colon X \to S$ is a K3 scheme. Indeed, if not, there exists an étale covering $S' \to S$ such that $X' = X \times_S S'$ is a K3 scheme over S'. Suppose that $\mathrm{Aut}_S(X')$ and $\mathrm{Aut}_S(X' \times_X X')$ are representable. Then the automorphisms of X over S correspond to the elements of $\mathrm{Aut}_S(X')$ satisfying the condition that the pullbacks for the two projections $p_1, p_2 : R = X' \times_X X' \to X'$ agree. Therefore, we can realize $\mathrm{Aut}_S(X)$ as the fiber product

$$\begin{array}{ccc} \mathrm{Aut}_S(X) & \longrightarrow & \mathrm{Aut}_S(X') \\ \downarrow & & \downarrow \\ \mathrm{Aut}_S(R) & \xrightarrow{\Delta} & \mathrm{Aut}_S(R) \times_S \mathrm{Aut}_S(R) \end{array}$$

so it is again representable by a scheme. The representability of $\mathrm{Aut}_S(X)$ for a K3 scheme by a group scheme locally of finite type over S follows from a more general result in [Gro62]. We need to check that $\mathrm{Aut}_S(X)$ is separated. By the valuative criterion, it amounts to showing that given a valuation ring R and two automorphisms $f_1, f_2 : X_R \to X_R$ which agree on the generic fiber X_K, then $f_1 = f_2$. Consider the morphism $(f_1, f_2) : X_R \to X_R \times_R X_R$; the points on which $f_1 = f_2$ are the preimage of the diagonal morphism Δ_{X_R}, which is closed by the separatedness of X_R. Hence, f_1 and f_2 agree on X_R. Finally, we want to prove that $\mathrm{Aut}_S(X)$ is unramified over S. In order to do so, since $\mathrm{Aut}_S(X) \to S$ is locally of finite type, it is enough to check ([Gro67], Ch.17) that the fibers over geometric points are discrete and reduced. Let $s : \mathrm{Spec}(k) \to S$ be a geometric point. The tangent space to the fiber of s at a point $t \colon k \to \mathrm{Aut}_k(X_k)$ can be identified with the points of $\mathrm{Aut}_k(X_k)(k[\varepsilon]/\varepsilon^2)$ over the point t chosen. These points give rise to vector fields on X_k. But since X_k is a K3 surface, we have $H^0(X_k, \mathcal{T}_{X_k}) = 0$, so the tangent space has dimension 0. This proves that the fibers of the morphism $Aut_S(X) \to S$ are discrete and reduced, so the morphism is unramified. $\square$

Suppose now that the K3 space $X \to S$ is endowed with a polarization λ. We can consider the automorphisms of the K3 space that respect the polarization

$$\mathrm{Aut}_S(X, \lambda)(T) = \{f \in \mathrm{Aut}_S(X) \mid f^*(\lambda) = \lambda\}.$$

The representability of the functor is preserved:

Proposition 5.1.6. *The functor* $\mathrm{Aut}_S(X,\lambda)$ *is representable by a separated group scheme which is unramified and of finite type over* S.

Proof. Define $\phi\colon \mathrm{Aut}_S(X)\times_S \mathrm{Pic}_{X/S} \to \mathrm{Pic}_{X/S}$ the group action of the automorphisms acting on line bundles via pullback. The functor $\mathrm{Aut}_S(X,\lambda)$ can be realized as a closed subfunctor of $\mathrm{Aut}_S(X)$, taking the fiber product in the diagram

$$\begin{array}{ccccc} \mathrm{Aut}_S(X,\lambda) & \longrightarrow & & \longrightarrow & S \\ \downarrow & & & & \downarrow \lambda \\ \mathrm{Aut}_S(X)\times_S S & \xrightarrow{(1,\lambda)} & \mathrm{Aut}_S(X)\times_S \mathrm{Pic}_{X/S} & \xrightarrow{\phi} & \mathrm{Pic}_{X/S} \end{array}$$

thus, $\mathrm{Aut}_S(X,\lambda)$ is representable by a separated scheme locally of finite type over S. The tangent space at a geometric point is again of relative dimension 0. In order to show that $\mathrm{Aut}_S(X,\lambda)$ is of finite type, it suffices to prove that the geometric fibers are finite. So, this reduces to showing that $\mathrm{Aut}_k(X_k,\lambda_k)$ is a finite set for an algebraically closed field k. Given an automorphism $f \in \mathrm{Aut}_k(X_k,\lambda_k)$ we can identify the morphism with its graph $\Gamma_f \subset X_k \times X_k$. The polarization λ_k provides an embedding of Γ_f in a suitable projective space, so that $\mathrm{Aut}_k(X_k,\lambda_k)$ can be considered as a subscheme of a Hilbert scheme. The fact that the relative dimension is 0 implies that it is finite, as Hilbert schemes are projective. □

5.1.2 Construction via Hilbert schemes

Hilbert schemes parametrize flat projective families over a certain scheme, with a given Hilbert polynomial. Consider the moduli functor $\mathbf{Hilb}^{N,P}$ that associates to a noetherian scheme S the families $\mathcal{Z} \to \mathbb{P}^N \times S$ with Hilbert polynomial P and flat over S. This functor is representable by a projective scheme over $\mathbb{Z}$, with a universal family $p : \mathcal{Z} \to \mathbf{Hilb}^{N,P_d}$ (see [Gro62], Exposé 221). For a K3 surface X a polarization of degree d is an ample line bundle $\mathcal{L}$. We know that $\mathcal{L}^3$ is ample, so it provides an embedding of X in $\mathbb{P}^N$, where $N = 9d + 1$. Moreover, the Hilbert polynomial corresponding to this embedding is given by $P_d(t) = 9dt^2 + 2$, by the Riemann-Roch formula for K3 surfaces, as in (2.3).

We wish to determine a subscheme of the Hilbert scheme $\mathbf{Hilb}^{N,P_d}$ which parametrizes projective K3 schemes.

Proposition 5.1.7. *There is a unique subscheme H of the Hilbert scheme $\mathbf{Hilb}^{N,P_d}$ such that a morphism of schemes $S \to \mathbf{Hilb}^{N,P_d}$ factors through H if and only if*

i) the pullback of the universal family $\mathcal{Z}_S \to S$ is a K3 scheme;

ii) there exist an ample line bundle $\mathcal{L}$ on $\mathcal{Z}_S$ and a line bundle $\mathcal{M}$ on S, such that

$$\mathcal{O}_{\mathcal{Z}_S}(1) = \mathcal{L}^3 \otimes p^*\mathcal{M};$$

iii) For every geometric point $s\colon \mathrm{Spec}(k) \to S$ the morphism $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathcal{Z}_s, \mathcal{L}_s^3)$ is an isomorphism.

In addition, there exists a unique open subscheme $H_P \subset H$ such that the morphism factors $S \to H$ factors through H_P if and only if the line bundle $\mathcal{L}_s$ is primitive at all geometric points s.

Proof. The argument follows the approach of [MFK65, Proposition 5.1], for the analogous statement for curves of a given genus. We are going to characterize the subscheme H, reducing the data to open or closed conditions on the basis.

i) The pullback of the universal family $\mathcal{Z}$ to H has to be a smooth proper morphism whose geometric fibers are K3 surfaces. The properness is automatic and being smooth is an open condition on the basis, so it determines an open subscheme $H_1 \subset \mathbf{Hilb}^{N,P_d}$, with universal family $\mathcal{Z}_1$. For the geometric fibers of morphisms $s\colon \mathrm{Spec}(k) \to H_1$, the vanishing of $H^1(\mathcal{Z}_s, \mathcal{O}_{\mathcal{Z}_s})$ is open by the upper semicontinuity result for flat, projective morphisms, so this determines an open subscheme $H_2 \subset H_1$. Finally, the isomorphism $O_{\mathcal{Z}_s} = \Omega^2_{\mathcal{Z}_s}$ is realized on the closed subscheme of H_2 obtained as the fiber product in the diagram

$$\begin{array}{ccc} H_3 & \longrightarrow & H_2 \\ \downarrow & & \downarrow {\scriptstyle (\mathcal{O}_{\mathcal{Z}_2}, \Omega_{\mathcal{Z}_2/H_2})} \\ \mathrm{Pic}_{\mathcal{Z}_2/H_2} & \xrightarrow{\Delta} & \mathrm{Pic}_{\mathcal{Z}_2/H_2} \times_{H_2} \mathrm{Pic}_{\mathcal{Z}_2/H_2} \end{array}$$

and so we get by pullback a universal family $\mathcal{Z}_3 \to H_3$ parametrizing projective flat K3 schemes with given Hilbert polynomial P_d.

ii) We can realize a closed suscheme H_4 of H_3 such that a family $S \to H_3$ factors through it if and only if there is an isomorphism $\mathcal{O}_{\mathcal{Z}_S}(1) = \mathcal{L}^3 \otimes p^*\mathcal{M}$ by taking the pullback in the diagram

$$\begin{array}{ccc} H_4 & \longrightarrow & H_3 \\ \downarrow & & \downarrow {\scriptstyle \mathcal{O}_{\mathcal{Z}_3}} \\ \mathrm{Pic}_{\mathcal{Z}_3/H_3} & \xrightarrow{[3]} & \mathrm{Pic}_{\mathcal{Z}_3/H_3} \end{array}$$

which gives a closed subscheme, since by Proposition 5.1.2 the morphism [3] is closed.

iii) Consider a family $\mathcal{Z}_S \to S$ factoring through H_4.The condition is equivalent to asking that for all geometric points $s\colon \mathrm{Spec}(k) \to S$ the fiber is not contained in any hyperplane. Although the condition is formulated depending on a choice of a line bundle $\mathcal{L}$ as in (ii), it is, in fact, independent of it. Let $p_1\colon \mathbb{P}_N \times S \to S$ be the projection on the first component. Since the cohomology $H^1(\mathcal{Z}_s, \mathcal{L}_S)$ vanishes, as the line bundle $\mathcal{L}_s$ is ample, the sheaf $p_*\mathcal{L}^3$ on S is locally free of rank $N+1$ (see [MFK65, Proposition 0.5]). There is a morphism $p_{1*}\mathcal{O}_{\mathbb{P}^N} \to p_*\mathcal{L}^3$ of locally free sheaves of rank $N+1$ over S. Denote the cokernel by $\mathcal{F}$. Taking the pullback on the geometric point s of S we obtain an exact sequence

$$p_{1*}\mathcal{O}_{\mathbb{P}^N} \otimes k \to p_*\mathcal{L}^3 \otimes k \to \mathcal{F} \otimes k \to 0$$

of vector spaces over k. Being $p_{1*}\mathcal{O}_{\mathbb{P}^N} \otimes k$ and $\mathcal{L}^3 \otimes k$ of dimension $N+1$, the first arrow is an isomorphism if and only if $\mathcal{F} \otimes k$ vanishes. But the map $p_{1*}\mathcal{O}_{\mathbb{P}^N} \otimes k \to p_*\mathcal{L}^3 \otimes k$ corresponds to the morphism $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathcal{Z}_s, \mathcal{L}_s^3)$. So the required condition holds on the subscheme of H_4 where the sheaf $\mathcal{F}$ vanishes, which determines a closed subscheme H with a universal family $\mathcal{Z}_H$ as in the statement. Finally , the line bundle $\mathcal{L}$ is primitive outside the closed subschemes S_k of H determined by the diagrams

$$\begin{array}{ccc} S_k & \longrightarrow & H \\ \downarrow & & \downarrow {\scriptstyle \mathcal{O}_{\mathcal{Z}_H}(1)} \\ \mathrm{Pic}_{\mathcal{Z}_H/H} & \xrightarrow{[k]} & \mathrm{Pic}_{\mathcal{Z}_H/H} \end{array}$$

for k^2 dividing d. This determines on open subscheme $H_P \subset H$ parametrizing primitively polarized $K3$ schemes as required.

□

5.2 Moduli space of polarized K3 surfaces

Let $\mathcal{S}$ be the category **Sch**. We define $\mathcal{M}_d$ to be the category of polarized K3 spaces over $\mathcal{S}$, whose objects are pairs $\mathcal{X} = (\pi\colon X \to S, \lambda)$, given by a K3 schemes with a polarization of degree d. The morphisms $\Phi \in \mathrm{Hom}(\mathcal{X}_1, \mathcal{X}_2)$ between objects

$$\mathcal{X}_1 = (\pi_1\colon X_1 \to S_1, \lambda_1), \qquad \mathcal{X}_2 = (\pi_2 : X_2 \to S_2, \lambda_2)$$

are pairs $\Phi = (\phi, \phi')$, where $\phi \colon S_1 \to S_2$ is a morphism of schemes and $\phi' \colon X_1 \to X_2 \times_{S_1} S_2$ is an isomorphism of schemes with $\phi'^* \lambda_2 = \lambda_1$. The functor $\mathcal{M}_d \to \mathcal{S}$

$$(\pi \colon X \to S, \lambda) \mapsto S$$

makes $\mathcal{M}_d$ a category fibered in groupoids over $\mathcal{S}$. Indeed, given a morphism of schemes $f \colon S_1 \to S_2$ and an object $\mathcal{X}_2 = (\pi_2 \colon X_2 \to S_2, \lambda_2)$ in the fiber of S_2 the fiber product $(X_2 \times_{S_2} S_1, f^*_{X_2}\lambda_2)$ is a pullback in $\mathcal{M}_d$. Moreover, for every scheme S, all the morphisms in the fiber $\mathcal{M}_d(S)$ are isomorphisms by definition.

In a similar way, we can define the full subcategory $\mathcal{P}_d$ of $\mathcal{M}_d$ of primitively polarized K3 schemes. The objects are $\mathcal{X} = (\pi \colon X \to S, \lambda)$ of K3 schemes with a primitive polarization of degree d. The proof for the representability of $\mathcal{P}_d$ and $\mathcal{M}_d$ as Deligne-Mumford stacks are the same, so we will just consider the case of $\mathcal{M}_d$ for simplicity.

Remark 5.2.1. In [Riz00], Rizov defines the moduli space $\mathcal{M}_d$ as the CFG having for objects polarized K3 algebraic spaces, rather than polarized K3 scheme. In fact, polarized varieties satisfy effectiveness of the descent. So, we can limit ourselves to objects in the category of schemes.

Proposition 5.2.2. *The category* $\mathcal{M}_d$ *is a stack over* $\mathcal{S}$.

Proof. We have to check that, for every pair of objects $\mathcal{X}$ and $\mathcal{Y}$ over a scheme S, the functor $\mathrm{Iso}_S(\mathcal{X}, \mathcal{Y})$ is a sheaf for the étale topology over S. This is verified because, as we will show later, this functor is, in fact, representable, and the étale topology is subcanonical.

Effectiveness of the descent: Let $f \colon S' \to S$ be an étale morphism, and let $\mathcal{X}' = (\pi' : X' \to S', \lambda')$ be an element in $\mathcal{M}_d(S')$ with a descent datum, that is an isomorphim between $\Phi \colon p_{1*}\mathcal{X} \to p_{2*}\mathcal{X}$, where $p_1, p_2 \colon S' \times_S S' \to S'$ are the first and second projections, satisfying the cocycle condition for the pullback on

$S''' = S' \times_S S' \times_S S'$ via the projections $p_{i,j} \colon S''' \to S''$

$$p_{1,2}^* \Phi = p_{1,3}^* \Phi \circ p_{3,2}^* \Phi.$$

Up to refining the étale covering $S' \to S$, we can assume that the polarization λ corresponds to the class of a relatively ample line bundle $\mathcal{L}$ in the Picard group. So we can assume that $\mathcal{L}^3$ defines a relatively very ample bundle for the morphism $\pi' \colon X' \to S'$. Let $\mathcal{S} = \mathrm{Sym}(\pi_* \mathcal{L})$ be the sheaf of symmetric algebras associated to $\pi'_* \mathcal{L}'$ over S'. Then the scheme X' corresponds to the relative projective space $\mathbf{Proj}(\mathcal{S}')$. The descent datum for $(\pi' : X' \to S', \lambda')$ induces a descent datum for the commutative graded algebra $\mathcal{S}$. Applying the descent for quasi-coherent sheaves of commutative algebras in [Vis08, Thm. 4.29], we can conclude that there exists a commutative graded algebra $\mathcal{S}$ over S such that $f^* \mathcal{S} = \mathcal{S}'$. Then, if we define $X = \mathbf{Proj}(\mathcal{S})$, the pullback along f of X is

$$X \times_S S' = \mathbf{Proj}(\mathcal{S}) \times_S S' = \mathbf{Proj}(\mathcal{S}') = X'.$$

Furthermore, we know that $\pi \colon X \to S$ is a K3 scheme, because this notion is local in the étale topology. Indeed, the fact that the morphism π is smooth and proper is étale local and the isomorphism $\mathcal{O}_{X_s} \simeq \Omega_{X_s}$ for the geometric fibers $s \in S$ is verified if and only if it is satisfied on an étale covering of S. So $X \to S$ is a K3 scheme. Now, by Prop. 5.1.2, we know that the Picard functor is, by definition a sheaf for the étale topology, therefore the descent datum for $\lambda' \in \mathrm{Pic}_{X/S}(S')$ glues to an element $\lambda \in \mathrm{Pic}_{X/S}(S)$ and defines again a polarization, as this condition is local in the étale topology. So we conclude that descent data are effective.

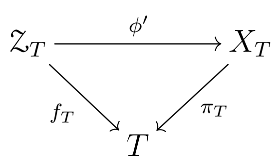
□

Proposition 5.2.3. *The category $\mathcal{M}_d$ is a separated algebraic stack over $\mathcal{S}$.*

Proof. The first condition for $\mathcal{M}_d$ to be an algebraic (Artin) stack is that the diagonal

$\Delta : \mathcal{M}_d \to \mathcal{M}_d \times_{\mathcal{S}} \mathcal{M}_d$ is representable by a separated algebraic space. This amounts to verifying that for every scheme S and every pair $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{M}_d(S)$ the scheme $\mathrm{Iso}_S(\mathcal{X}, \mathcal{Y})$ satisfies the same properties. It is enough to verify this for $\mathrm{Aut}_S(\mathcal{X})$, in which case it follows from Prop. 5.1.6.

Then we have to show that $\mathcal{M}_d$ admits a smooth atlas. Indeed, we can define a morphism $H \to \mathcal{M}_d$, where H is the subscheme of the Hilbert scheme defined in Proposition 5.1.7, associating to each family $\mathcal{Z} \to \mathbb{P}_N \times S$, parametrized by the morphism $S \to H$, the K3 scheme $(f_S \colon \mathcal{Z}_S \to S, \lambda)$ where $\lambda = [\mathcal{L}]$ is the class of the line bundle $\mathcal{L}$ such that $[\mathcal{L}^3] = [\mathcal{O}_{\mathcal{Z}_S}(1)]$ in the Picard group $\mathrm{Pic}_{\mathcal{Z}_H/H}(S)$. We want to prove that this morphism $H \to \mathcal{M}_d$ is representable by a smooth scheme. The representability follows from the fact that the diagonal $\Delta \colon \mathcal{M}_d \to \mathcal{M}_d \times_{\mathcal{S}} \mathcal{M}_d$ is representable. Let S be a scheme and $S \to \mathcal{M}_d$ a morphism defined by the element $(\pi \colon X \to S, \mu)$. Passing to an étale covering, we can suppose that $X \to S$ is a K3 scheme, that the class μ corresponds to a line bundle $\mathcal{E}$ and, refining the covering further, we can assume that the line bundle $\mathcal{E}^3$ is very ample and $\pi_* \mathcal{E}$ is a free sheaf on S. Therefore $\mathcal{E}$ induces a morphism $X \to \mathbb{P}(\pi_* \mathcal{E}^3) = \mathbb{P}^N \times S$. For a scheme T, the elements of the fiber product $(H \times_{\mathcal{M}_d} S)(T)$ can be identified with the triple $(T \to S, T \to H, \Phi)$, where $\Phi = (\mathrm{id}_T, \phi')$ is defined by an isomorphism



such that ${\phi'}^*(\mu_T) = \lambda_T$ in $\mathrm{Pic}_{\mathcal{Z}/H}(T)$. Hence we have an equality

$$(\phi'^* \mathcal{E})^3 = \mathcal{O}_{\mathcal{Z}_T}(1)^3 \otimes f_T^* \mathcal{M}$$

of sheaves on $\mathcal{Z}_T$, for some line bundle $\mathcal{M}$ over T. For every line bundle over $\mathcal{Z}_T$ such that $\mathcal{L}^3 = \mathcal{O}_{\mathcal{Z}_T}(1)^3 \otimes f_T^* \mathcal{M}$ we have an isomorphism $f_{T*}(\mathcal{L}^3) = p_{1*} \mathcal{O}_{\mathbb{P}^N}(1)$, so in particular $f_{T*}(\phi'^{*}\mathcal{E}^3) = p_{1*}\mathcal{O}_{\mathbb{P}^N}(1)$. Hence, we have

$$\mathbb{P}(f_{T*}(\phi'^{*}\mathcal{E}^3)) = \mathbb{P}(p_{1*}\mathcal{O}_{\mathbb{P}^N}(1)) = \mathbb{P}_N \times T.$$

So the isomorphism $\phi' : \mathcal{Z}_T \to X_T$ is, in fact, induced by an automorphism of $\mathbb{P}^N \times T$ over T. Therefore, we can identify $(H \times_{\mathcal{M}_d} S)(T) = \mathrm{PGL}(N+1)(T)$. As the scheme $\mathrm{PGL}(N+1)$, being a group scheme, is smooth, we can conclude that $\mathcal{M}_d$ is an algebraic stack. □

Theorem 5.2.4. *The category $\mathcal{M}_d$ is a Deligne-Mumford stack.*

Proof. We can use the criterion in [LMB00], Thm. 8.1, that an algebraic stack is a Deligne-Mumford stack if and only if the diagonal $\Delta : \mathcal{M}_d \to \mathcal{M}_d \times_{\mathcal{S}} \mathcal{M}_D$ is unramified. Once again, this amounts to checking that for every scheme S and every two elements $\mathcal{X}$ and $\mathcal{Y}$ in the fiber $\mathcal{M}_d(S)$ the morphism $\mathrm{Iso}_S(\mathcal{X}, \mathcal{Y}) \to S$ is unramified, which follows from Proposition 5.1.6. □

5.3 Moduli space of polarized K3 surfaces with level structure

We have shown that the moduli space of polarized K3 surfaces with level structures is realized as a Deligne-Mumford stack. The same result holds for the moduli space of polarized abelian $\mathcal{A}_{g,d'}$. The analogy with the case of abelian varieties suggests to construct the moduli space of polarized K3 surfaces with a level structure. For an abelian variety A, the datum of a level N structure, that is an isomorphism between the N-torsion points $A[N]$ and the standard symplectic lattice over $\mathbb{Z}/N\mathbb{Z}$, can be interpreted as a choice of an isomorphism between the Tate module $T_\ell A$ and a symplectic module over $\mathbb{Z}_\ell$, up to the action of a subgroup $\mathbb{K}_\ell \subset \mathrm{CSp}(\mathbb{Z}_\ell)$. A similar interpretation can be generalized to K3 surfaces. As for abelian varieties, the CFG of K3 surfaces with level structures turns out to be an algebraic space, if the level is sufficiently fine.

5.3.1 Level structures

Let $(X \to S, \lambda)$ be a K3 scheme with a primitive polarization of type d, and suppose that S is connected. For every geometric point $s\colon \mathrm{Spec}(k) \to S$, the polarization λ induces an isometry

$$\Lambda_{d,\mathbb{Z}_\ell} \simeq PH^2(X_s, \mathbb{Z}_\ell)$$

as abstract lattices, for all ℓ coprime to the characteristic of k. Let $\mathbb{K} = (\mathbb{K}_\ell)$ be an open compact subgroup of $\mathrm{SO}(\Lambda_{d,\mathbb{Q}})(\hat{\mathbb{Z}})$ of finite index.

Definition 5.3.1. A $\mathbb{K}$-level structure on the polarized K3 scheme $(X \to S, \lambda)$ is a class α_ℓ in

$$\mathbb{K}_\ell \backslash \mathrm{Isom}(\Lambda_{d,\mathbb{Z}_\ell}, PH^2(X_s, \mathbb{Z}_\ell)),$$

for the action of $\mathbb{K}_\ell$ on $\Lambda_{d,\mathbb{Z}_\ell}$, fixed by the monodromy action of $\pi_1^{\mathrm{alg}}(S, s)$ on $PH^2(X, \mathbb{Z}_\ell)$, for all ℓ coprime with all residual characteristics of S. More generally, a $\mathbb{K}$-level structure for a K3 scheme over an arbitrary scheme S is the datum of a level structure over every connected component.

The datum of a $\mathbb{K}$-level structure for a geometric point $s \in S$ induces a similar structure for every other geometric point $t \in S$, since there is an isomorphism $\pi_1^{\mathrm{alg}}(S, s) \simeq \pi_1^{\mathrm{alg}}(S, t)$, unique up to inner automorphism, inducing an isometry $PH^2(X_s, \mathbb{Z}_\ell) \simeq PH^2(X_t, \mathbb{Z}_\ell)$. If α is a $\mathbb{K}$-level structure on $(X \to S, \lambda)$, it induces an isomorphism $\alpha'\colon O(PH^2(X_s, \mathbb{Z}_\ell)) \to O(\Lambda_{d,\mathbb{Z}_\ell})$, such that the monodromy $\pi_1^{\mathrm{alg}}(S, s) \to O(PH^2(X_s, \mathbb{Z}_\ell))$ factors through $\alpha^{-1}(\mathbb{K}_\ell)$. Since $\mathbb{K}$ is a subgroup of finite index in $O(\Lambda_d)(\hat{\mathbb{Z}})$, for every polarized K3 scheme $X \to S$ we can pass to an étale covering $S' \to S$ such that the monodromy factors through it, so that it admits a $\mathbb{K}$-level structure.

Example 5.3.2. Let $N > 0$ be an integer. We can define the group $\mathbb{K}_N = (\Gamma_{N,\ell})$ as in

(4.2). Then the datum of a $\mathbb{K}_N$-structure for a K3 scheme $(\pi\colon X \to S, \lambda)$ corresponds to the choice of an isomorphism $P^2\pi_*(X, \mathbb{Z}/N\mathbb{Z}) \simeq \Lambda_{d,\mathbb{Z}/N\mathbb{Z}}$, where the right hand side is the constant sheaf over S. Another important example is the group $\mathbb{K}_N^{\mathrm{ad}} = (\Gamma_N^{\mathrm{ad}})$, which is obtained as the isomorphic image via the adjoint morphism of

$$(\Gamma_{N,\ell}^{\mathrm{sp}}) = \mathbb{K}_N^{\mathrm{sp}} \subset \mathrm{CSpin}(\Lambda_d \otimes \mathbb{Q})(\hat{\mathbb{Z}})$$

as in (4.1). We call this the spin level N structure.

5.3.2 Moduli space of polarized K3 surfaces with level structures

Let $\mathbb{K} \subset SO(\Lambda_{d,\mathbb{Q}})(\hat{\mathbb{Z}})$ be a subgroup of finite index, such that $\mathbb{K} \subset \mathbb{K}_N$ for some N, with $N > 3$, and assume N coprime to $2d$. Consider the category fibered in groupoids $\mathcal{P}_{d,\mathbb{K}}$ over the category $\mathcal{S} = \mathbf{Sch}/\mathbb{Z}[1/N]$, whose objects are triples $\mathcal{X} = (\pi\colon X \to S, \lambda, \alpha)$ of polarized K3 schemes with a level structure. Given two objects

$$\mathcal{X}_1 = (\pi_1\colon X_1 \to S_1, \lambda_1, \alpha_1) \qquad \mathcal{X}_2 = (\pi\colon X_2 \to S_2, \lambda_2, \alpha_2)$$

consider the pair $\Phi = (\phi, \phi')$, where $\phi : S_1 \to S_2$ is a morphism of schemes and $\phi' : X_1 \to X_2 \times_{S_1} S_2$ is an isomorphism of schemes with $\phi'^*\lambda_2 = \lambda_1$. For a geometric point $s_1 \in S_1$ and a prime ℓ we have an isomorphism $\phi'^*\colon PH^2(X_{\phi(s_1)}, \mathbb{Z}_\ell) \to PH^2(X_{s_1}, \mathbb{Z}_\ell)$. This induces a class $\hat{\phi}(\alpha_2) \in \mathbb{K}_\ell \setminus \mathrm{Isom}(\Lambda_{d,\mathbb{Z}_\ell}, PH^2(X_{s_1}, \mathbb{Z}_\ell)$. We define an element of $\Phi \in \mathrm{Hom}_{\mathcal{P}_{d,\mathcal{K}}}(\mathcal{X}_1, \mathcal{X}_2)$ to be a pair $\Phi = (\phi, \phi')$ as before, with the additional condition that $\hat{\phi}(\alpha_2) = \alpha_1$.

Proposition 5.3.3. *A triple $\mathcal{X} = (\pi\colon X \to S, \lambda, \alpha)$ has only trivial automorphisms.*

Proof. Since $\mathbb{K} \subset \mathbb{K}_N$, it is enough to prove the proposition for $\mathbb{K}_N$. Let ℓ be a prime such that $\ell^r \mid\mid N$. An automorphism of the triple associated to $\mathcal{X}$ is given by an automorphism ϕ' of X over S fixing λ. Let $s \in S$ a geometric point. This induces an automorphism g of $PH^2(X_s, \mathbb{Z}_\ell)$ of finite order, as $\mathrm{Aut}(X_s, \lambda_s)$ is of finite order. Therefore, g is semisimple with eigenvalues given by roots of unity. The condition that $\hat{\phi}(\alpha_2) = \alpha_1$ implies that $g \equiv 1 \pmod{\ell^r}$. Then, if ω is an eigenvalue for g, it is a root of unity satisfying the relation

$$\omega = 1 + \ell^r \eta$$

for some algebraic integer η. From this we can conclude that $\eta = 0$ as in [Mum74, Ch. VI, Appl. II, p. 207]. The automorphism ϕ'^* fixes the λ_s and is trivial on the primitive part of the cohomology $PH^2(X_s, \mathbb{Z}_\ell)$, so it is the identity on the whole cohomology. As $\mathrm{Aut}(X_s, \lambda)$ acts faithfully on $H^2(X_s, \mathbb{Z}_\ell)$, we can conclude that the automorphism ϕ_s^* of the fiber X_s is trivial. The same argument works for all fibers, so ϕ' is trivial. □

The fact that the objects in the category $\mathcal{P}_{d,\mathbb{K}}$ have no non-trivial automorphisms is the key step in showing its representability as an algebraic space.

Theorem 5.3.4. *The category $\mathcal{P}_{d,\mathbb{K}}$ is a separated algebraic space over $\mathbb{Z}[1/N]$.*

Proof. The fact that $\mathcal{P}_{d,\mathbb{K}}$ is a stack can be proven with the same argument as for $\mathcal{P}_d$. In order to show that $\mathcal{P}_{d,\mathbb{K}}$ is a Deligne-Mumford stack, the representability of the diagonal follows from Prop. 5.3.3. An étale covering can be obtained by taking an étale covering U of $\mathcal{P}_d$ and refining it to make the monodromy factor through $\mathbb{K}$. Then, as objects in this category have only trivial automorphisms, the CFG is fibered in **Set**. As it is a Deligne-Mumford stack, it is, in fact, an algebraic space. The separatedness follows from the valuative criterion as in the case of $\mathcal{P}_d$. □

Remark 5.3.5. We conclude by listing some properties of the moduli functor for K3 surfaces. The moduli spaces of polarized K3 surfaces $\mathcal{M}_d$ and $\mathcal{P}_d$ are smooth over $\mathbb{Z}[1/2d]$. In fact, a result by Keel and Mori (see [KM97]) states that the existence of the moduli space as a Deligne-Mumford stack implies the existence of a coarse moduli space for $\mathcal{M}_d$ and $\mathcal{P}_d$ given by an algebraic space. Via Geometric Invariant Theory techniques, one can prove that, in characteristic zero, this algebraic space is a quasi-projective scheme. Similarly, the algebraic space $\mathcal{P}_{d,\mathbb{K}}$ is smooth over $\mathbb{Z}[1/2dN]$. Therefore, it will be natural to consider these restrictions in the following.

Chapter 6

The Kuga-Satake map over moduli spaces

6.1 The Kuga-Satake map in characteristic zero

In order to describe a Kuga-Satake morphism with the machinery of moduli spaces, we can try to interpret the classical Kuga-Satake construction we carried out in Sect. 3.2 in terms of Shimura varieties, as they parametrize polarized Hodge structures of a suitable kind. The existence of canonical models for Shimura varieties makes the construction descend over a number field.

6.1.1 Interpretation via Shimura varieties

Let $(V_{\mathbb{Z}}, Q)$ be a free $\mathbb{Z}$-module with a quadratic form and denote by $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$. We will consider the case where $(V_{\mathbb{Z}}, Q) = (\Lambda_d, \psi_d)$, for the standard lattice with quadratic form ψ_d of signature $(2-, 19+)$. Let $C = C^+(V_{\mathbb{Z}})$ be the even Clifford algebra associated to $(V_{\mathbb{Z}}, Q)$. Define $L_{\mathbb{Z}}$ a free module of rank 1 over C, $L = L_{\mathbb{Z}} \otimes \mathbb{Q}$, and let $a \in C$ be an element satisfying $\iota(a) = -a$. Denote by G the group $\mathrm{CSpin}(V)$. Sending an element $g \in G$ to the left multiplication by g in L defines an injective morphism, that we denote by sp,

$$\mathrm{sp}\colon G \to \mathrm{GSp}(L, \phi_a),$$

where $\mathrm{GSp}(L, \phi_a)$ is the group of symplectic similitudes of L with respect to the form ϕ_a defined as in Prop. 3.2.4. Indeed, if $x, y \in L$, and $g \in G$, we have

$$\phi_a(gx, gy) = \mathrm{Tr}(\iota(gx)gya) = N(g)\mathrm{Tr}(\iota(x)ya) = \phi_a(x, y).$$

For a morphism $h_0\colon \mathbb{S} \to G_{\mathbb{R}}$, the composition $\mathrm{sp} \circ h_0$ defines a Hodge structure of weight 1 on L, for which ϕ_a is a polarization of some degree d'^2 and type $(\delta_1, \delta_2, \dots \delta_n)$, for $n = 2^{19}$, depending on the element a chosen. This determines a morphism of Shimura data

$$\mathrm{sp}\colon (G, \Omega^{\pm}) \to (\mathrm{GSp}(L, \phi_a), \mathfrak{H}_n^{\pm}).$$

If we define

$$\Lambda_N = \{x \in \mathrm{GSp}(L, \phi_a) \mid x \equiv 1 \mod n\}$$

the image of $\mathbb{K}_N^{sp}$ under the morphism sp is contained in Λ_N. Therefore, the morphism of Shimura data determines a map

$$\mathbf{sp}\colon \mathbf{Sh}_{\mathbb{K}_N^{\mathrm{sp}}}(G, \Omega^{\pm}) \to \mathbf{Sh}_{\Lambda_N}(\mathrm{GSp}(L, \phi_a), \mathfrak{H}_n^{\pm}),$$

which characterizes the spinorial Shimura variety $\mathbf{Sh}_{\mathbb{K}_N^{\mathrm{sp}}}(G, \Omega^{\pm})$ as a Shimura variety of Hodge type, with reflex field $\mathbb{Q}$ (see [And96], Appendix A). Such Shimura varieties parametrize polarized abelian varieties with a level structure and a fixed set of Hodge tensors, as proven by Deligne in [Del82]. Thus, the spinorial Shimura variety provides a characterization of the image of the Kuga-Satake map.

As the Kuga-Satake map is defined by taking a polarized Hodge structure of K3 type $h_0\colon \mathbb{S} \to \mathrm{SO}(V_{\mathbb{R}})$ and sending it to

$$\mathrm{sp} \circ \tilde{h}_0 \colon \mathbb{S} \to \mathrm{GSp}(L, \phi_a),$$

we wish to interpret it as a morphism from the corresponding Shimura variety of orthogonal type into the Siegel modular variety $\mathbf{Sh}_{\Lambda_N}(\mathrm{GSp}(L, \phi_a), \mathfrak{H}_n^{\pm})$. Denote $G^{\mathrm{ad}} = \mathrm{SO}(V)$ the special orthogonal group. The morphism $\mathrm{ad}\colon G \to G^{\mathrm{ad}}$, induces a morphism of Shimura data, denoted again by ad,

$$\mathrm{ad}\colon (G, \Omega^{\pm}) \to (G^{\mathrm{ad}}, \Omega^{\pm}).$$

For $N \geq 3$, the isomorphic image of $\mathbb{K}_N^{\mathrm{sp}}$ is $\mathbb{K}_N^{\mathrm{ad}}$. This yields a morphism of Shimura varieties, defined over $\mathbb{Q}$,

$$\mathbf{ad}\colon \mathbf{Sh}_{\mathbb{K}_N^{\mathrm{sp}}}(G, \Omega^{\pm}) \to \mathbf{Sh}_{\mathbb{K}_N^{\mathrm{ad}}}(G^{\mathrm{ad}}, \Omega^{\pm}).$$

Then, in order to define a Kuga-Satake map, it is enough to determine a section γ of this morphism. The complex points of these Shimura varieties can be identified with

$$\mathbf{Sh}_{\mathbb{K}_N^{\mathrm{sp}}}(G, \Omega^{\pm})(\mathbb{C}) = G(\mathbb{Q})\backslash\Omega^{\pm} \times G(\mathbb{A}_f)/\mathbb{K}_N^{\mathrm{sp}}.$$

and

$$\mathbf{Sh}_{\mathbb{K}_N^{\mathrm{ad}}}(G^{\mathrm{ad}}, \Omega^{\pm})(\mathbb{C}) = G^{\mathrm{ad}}(\mathbb{Q})\backslash\Omega^{\pm} \times G^{\mathrm{ad}}(\mathbb{A}_f)/\mathbb{K}_N^{\mathrm{ad}}$$

respectively. Furthermore, applying a general result in [Mil04, Lemma 5.13], they break down into connected components as

$$\mathbf{Sh}_{\mathbb{K}}(G{\cdot}\Omega^{\pm}) = \bigsqcup_{g \in \mathcal{C}} \Gamma_{[g]}\backslash\Omega^{\pm} \qquad \text{and} \qquad \mathbf{Sh}_{\mathbb{K}}(G^{\mathrm{ad}}, \Omega^{\pm}) = \bigsqcup_{h \in \mathcal{C}^{\mathrm{ad}}} \Gamma_{[h]}\backslash\Omega^{\pm}, \qquad (6.1)$$

where the disjoint unions run over sets of representatives $\mathcal{C}$ and $\mathcal{C}_{ad}$ of classes in

$G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/\mathbb{K}_N^{\text{sp}}$ and $G^{\text{ad}}(\mathbb{Q})_+\backslash G^{\text{ad}}(\mathbb{A}_f)/\mathbb{K}_N^{\text{ad}}$. The embedding $\Gamma_{[g]}\backslash\Omega^\pm \hookrightarrow \mathbf{Sh}_{\mathbb{K}}(G^{\text{ad}},\Omega^\pm)$ is given by $[x]\mapsto[x,g]$. On each connected component the morphism **ad** corresponds to

$$\Gamma_{[g]}\backslash\Omega^\pm \to \Gamma_{[\text{ad}(g)]}\backslash\Omega^\pm$$
$$[x,g]\mapsto[x,\text{ad}(g)]$$

which defines an isomorphism of complex manifolds, because the groups $\Gamma_{[g]} = G_+(\mathbb{Q})\cap g\mathbb{K}_N^{\text{sp}}g^{-1}$ and $\Gamma_{[\text{ad}(g)]} = G_+^{\text{ad}}(\mathbb{Q})\cap \text{ad}(g)\mathbb{K}_N^{\text{ad}}\text{ad}(g)^{-1}$ are isomorphic. In order to define a section γ of **ad**, it suffices to determine a section of the induces morphism on the connected components. Via class field theory, using the Artin map, we can identify the sets of representatives as

$$\mathcal{C} = \text{Gal}(E/\mathbb{Q}) \qquad\qquad \mathcal{C}^{\text{ad}} = \text{Gal}(E'/\mathbb{Q})$$

for two abelian extension $E'\subset E$ of $\mathbb{Q}$. More precisely, the set of connected components can explicitly be determined as

$$\begin{aligned}\pi_0(\mathbf{Sh}_{\mathbb{K}_N^{\text{sp}}}(G,\Omega^\pm)) &= G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/\mathbb{K}_N^{\text{sp}}\\ &= \mathbb{G}_m(\mathbb{A})/(\mathbb{Q}^*\mathbb{R}_{>0}N(\mathbb{K}_N^{\text{sp}})).\end{aligned}$$

The subgroup $\mathbb{Q}^*\mathbb{R}_{>0}N(\mathbb{K}_N^{\text{sp}})\subset\mathbb{G}_m(A)$ is isomorphic to the Galois $\text{Gal}(E/\mathbb{Q})$, where E is a finite abelian extension of $\mathbb{Q}$. Hence we have that

$$\text{art}\colon \mathbb{G}_m(\mathbb{A})/(\mathbb{Q}^*\mathbb{R}_{>0}N(\mathbb{K}_N^{\text{sp}}))\to\text{Gal}(E/\mathbb{Q})$$

is an isomorphism. Then, the set of connected components can be identified with $\text{Gal}(E/\mathbb{Q})$, and this group acts freely on $\pi_0(\mathbf{Sh}_{\mathbb{K}_N^{\text{sp}}}(G,\Omega^\pm)$, so each connected component is defined over E. Similarly, for the orthogonal Shimura variety $\mathbf{Sh}_{\mathbb{K}_N^{\text{ad}}}(G^{\text{ad}},\Omega^\pm)$

the set of connected components can be identified with $\mathrm{Gal}(E'/\mathbb{Q})$, where $E' \subset E$ is the field of definition of each connected component.

To give a section of the morphism ad amounts to giving a section of the morphism $\mathrm{Gal}(E/\mathbb{Q}) \to \mathrm{Gal}(E'/\mathbb{Q})$ (as morphism of sets). The choice of such a morphism determines a map

$$\mathbf{sp} \circ \gamma \colon \mathbf{Sh}_{\mathbb{K}_N^{\mathrm{ad}}}(G^{\mathrm{ad}}, \Omega^{\pm}) \to \mathbf{Sh}_{\Lambda_N}(\mathrm{GSp}(L, \phi_a), \mathfrak{H}_n^{\pm})$$

over the field E, which provides an interpretation via Shimura varieties of the classical Kuga-Satake construction.

6.1.2 The period mapping

To define a Kuga-Satake map on the moduli space of K3 surfaces with $\mathbb{K}$-level structure, where $\mathbb{K}$ is a subgroup of finite index in $\mathbb{K}_N$, we need to relate the moduli space $\mathcal{P}_{d,\mathbb{K}}$ to the orthogonal Shimura varieties previously defined. In other words, we have to determine a period mapping. This result can be achieved exploiting the modular interpretation of the Shimura variety $\mathbf{Sh}_{\mathbb{K}}(G^{\mathrm{ad}}, \Omega^{\pm})$.

As in the case of the Siegel modular variety (see [Mil04, Prop. 6.3]), the complex points of this Shimura variety can be described as isomorphism classes $[W, h, s, \alpha]$ where W is a $\mathbb{Q}$-vector space, h a Hodge structure for W, s a polarization for h and α is the class of an isogeny in

$$\mathbb{K}\backslash \mathrm{Isog}((\Lambda_d \otimes \mathbb{A}_f, Q), (W \otimes \mathbb{A}_f, s)).$$

By an isogeny, here, we mean a morphism of $\mathbb{A}_f$-modules, which is an isometry for the quadratic forms Q and s up to multiplication by a scalar. Given an isomorphism class $[W, h, s, \alpha]$, one can choose an isometry $\phi \colon (W, s) \to (V, Q)$, which exists by the Hasse-Minkowski principle. This determines an isomorphism $\mathrm{SO}(W) \to G^{\mathrm{ad}}$ taking

$g \mapsto \phi \circ g \circ \phi^{-1}$, so we can associate to the isomorphism class $[W, h, s, \alpha]$ the complex point of the Shimura variety

$$[\phi \circ h \circ \phi^{-1}, \phi \circ \tilde{\alpha}],$$

for a representative $\tilde{\alpha}$ of α. Thanks to this characterization, it is natural to define a period mapping $j(\mathbb{C})\colon \mathcal{P}_{d,\mathbb{K}}(\mathbb{C}) \to \mathbf{Sh}_{\mathbb{K}}(G^{\mathrm{ad}}, \Omega^{\pm})(\mathbb{C})$. Indeed, given a triple (X, λ, α) of a complex K3 surface with polarization and Hodge $\mathbb{K}$-level structure, we can define its period to be

$$j(\mathbb{C})((X, \lambda, \alpha)) = [PH^2(X, \mathbb{Q})(1), (\ ,\ ), \alpha]$$

where $(\ ,\ )$ is the polarization induced by the intersection pairing on the primitive part of the cohomology with respect to the Chern class of the line bundle λ. We want to show that this pointwise definition of a period morphism extends to a map of algebraic spaces.

Theorem 6.1.1. *There is an étale morphism of algebraic spaces over* $\mathbb{C}$

$$j\colon \mathcal{P}_{d,\mathbb{K},\mathbb{C}} \to \mathbf{Sh}_{\mathbb{K}}(G^{\mathrm{ad}}, \Omega^{\pm})_{\mathbb{C}}$$

such that on the complex points it agrees with $j(\mathbb{C})$.

Proof. The separated algebraic space $\mathcal{P}_{d,\mathbb{K},\mathbb{C}}$ admits an étale covering by a scheme U over $\mathbb{C}$, with a universal family $(\pi_U\colon X_U \to U, \lambda_U, \alpha_U)$. Replace U with one of its algebraic components. The level structure α is the $\mathbb{K}$-class of an isometry

$$\tilde{\alpha}_u\colon \Lambda_{d,\hat{\mathbb{Z}}} \to PH^2(X_u, \hat{\mathbb{Z}})$$

for a point $u\colon \mathrm{Spec}(\mathbb{C}) \to U$, that we can suppose fixed. We begin performing an analytic construction and then showing that it is, in fact, algebraic. Consider the analytification $X_U^{\mathrm{an}} \to U^{\mathrm{an}}$ of the universal K3 scheme, with the polarization

λ_U. For every point $v \in U^{\mathrm{an}}$ we can define the period to be $j(\mathbb{C})(\mathcal{X}_v)$, where $\mathcal{X}_v = (X_v, \lambda_v, \alpha_v)$. We want to see that this morphism maps U^{an} into a connected component of $\mathbf{Sh}_{\mathbb{K}}(G^{\mathrm{ad}}, \Omega^{\pm})$. The choice of an isomorphism $\xi\colon \pi_1(U^{\mathrm{an}}, u) \to \pi_1(U^{\mathrm{an}}, v)$ induces an isometry $\xi'\colon PH^2(X_u, \mathbb{Z}) \to PH^2(X_v, \mathbb{Z})$ for the intersection form. As we have seen in Sect. 5.3.1, the class α_v is defined as the $\mathbb{K}$-class of $\xi'_{\hat{\mathbb{Z}}} \circ \tilde{\alpha}_u$. So, the periods of $\mathcal{X}_u$ and $\mathcal{X}_v$ can be expressed as

$$j(\mathbb{C})(\mathcal{X}_u) = [\phi \circ h_u \circ \phi^{-1}, \phi \circ \tilde{\alpha}_u] \qquad j(\mathbb{C})(\mathcal{X}_u) = [\phi \circ \xi'^{-1} \circ h_v \circ \xi' \circ \phi^{-1}, \phi \circ \tilde{\alpha}_u]$$

for the choice of an isometry $\phi\colon (PH^2(X_u, \mathbb{Z}), \psi_{\lambda_u}) \to (\Lambda_d, \psi_d)$. This shows that U is mapped into the connected component $\Gamma_{[g]}\backslash\Omega^{\pm}$ of the Shimura variety for $g = \phi \circ \tilde{\alpha}_u$ in the decomposition (6.1).

The sheaf $P^2\pi_{U*}\mathbb{Z}$ is locally constant and defines a holomorphic Hodge variation on U^{an}. Hence, locally on the transcendental topology we can find a connected an open neighborhood $\tilde{U}$ of u such that, via the choice of the marking ϕ, we can define a holomorphic map $\tilde{\jmath}\colon \tilde{U} \to \Omega^{\pm}$. This yields to a diagram

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{\jmath}} & \Omega^{\pm} \\ \downarrow & & \downarrow \\ U & \xrightarrow{j} & \Gamma_{[g]}\backslash\Omega^{\pm} \end{array}$$

where the map $\tilde{\jmath}$ is a local isomorphism by the local Torelli Theorem (see [Huy, Chap. 6, Prop. 2.8]) and the projection $\Omega \to \Gamma_{[g]}\backslash\Omega^{\pm}$ is a covering. This proves that j is a holomorphic map of analytic spaces and a local isomorphism. As it is holomorphic, since $\Gamma_{[g]}$ is torsion-free, j is also algebraic by Baily-Borel theorem. Being an analytic local isomorphism, j is étale. It remains to show that j descends to a map $j\colon \mathcal{P}_{d,\mathbb{K},\mathbb{C}} \to \mathbf{Sh}_{\mathbb{K}}(G^{\mathrm{ad}}, \Omega^{\pm})_{\mathbb{C}}$. In order to prove this, one should prove that the two morphism

$$U \times_{\mathcal{P}_{d,\mathbb{K},\mathbb{C}}} U \to \mathbf{Sh}_{\mathbb{K}}(G^{\mathrm{ad}}, \Omega^{\pm})$$

induced by the projections $p_1, p_2 \colon U \times_{\mathcal{P}_{d,\mathbb{K},\mathbb{C}}} U \to U$ agree. As $\mathcal{P}_{d,\mathbb{K},\mathbb{C}}$ is reduced, it is enough to verify the equality on $\mathbb{C}$-valued points, which is trivial. □

Combining this result with [Knu71, Ch. II, Cor. 6.17], we see that the algebraic space $\mathcal{P}_{d,\mathbb{K},\mathbb{C}}$ is a scheme.

6.1.3 The Kuga-Satake morphism in characteristic zero

Let $\mathcal{A}_{n,d',N}$ be the moduli space of abelian varieties of genus n, with a polarization of degree d'^2, and level $N \geq 3$. This is a separated algebraic space over $\mathbb{Z}[1/N]$, smooth over $\mathbb{Z}[1/Nd']$. In fact, Mumford proves in [MFK65, Chap. 7], that $\mathcal{A}_{n,d',N}$ is represented by a quasi-projective scheme, which provides a canonical model for the Siegel modular variety over $\mathbb{Q}$, as shown in [Mil92, Thm. 2.10]. More precisely, the choice of $a \in C$ as before, induces a symplectic form of type $(\delta_1, \delta_2, \ldots, \delta_n)$, with $d' = \delta_1 \cdot \delta_2 \cdots \delta_n$, on L. This let us define a morphism

$$i \colon \mathbf{Sh}_{\Lambda_N}(\mathrm{GSp}(L, \phi_a), \mathfrak{H}_n^{\pm}) \to \mathcal{A}_{n,d',N,\mathbb{Q}}$$

that identifies the Siegel modular variety with a component of the moduli space of abelian varieties with level structure and a polarization of degree d'^2. We are now ready to state the main result of this section.

Theorem 6.1.2 (Kuga-Satake in characteristic zero)**.** *There exists a morphism of algebraic spaces over E*

$$\mathrm{KS} \colon \mathcal{P}_{d,\mathbb{K}_N^{sp},E} \to \mathcal{A}_{n,d,N,E}$$

sending a polarized K3 surface with spin level N structure to its Kuga-Satake variety, with polarization corresponding to $a \in L$ and level N structure.

Proof. We have seen that we can interpret the classical Kuga-Satake construction as a morphism of Shimura varieties,

$$\mathbf{sp} \circ \gamma \colon \mathbf{Sh}_{\mathbb{K}_N^{\mathrm{ad}}}(G^{\mathrm{ad}}, \Omega^{\pm}) \to \mathbf{Sh}_{\Lambda_N}(\mathrm{GSp}(L_{\mathbb{Q}}, \phi_a), \mathfrak{H}_n^{\pm}).$$

defined over the number field E. On the other hand, the period morphism provides a map

$$j \colon \mathcal{P}_{d,\mathbb{K},\mathbb{C}} \to \mathbf{Sh}_{\mathbb{K}}(G^{\mathrm{ad}}, \Omega^{\pm})_{\mathbb{C}}$$

defined over $\mathbb{C}$. Rizov shows in [Riz05] that the map j descends to $\mathbb{Q}$, exploiting an analog of the main theorem of complex multiplication for K3 surfaces. Thus, we can let the Kuga-Satake map be the composition

$$i \circ \mathbf{sp} \circ \gamma \circ j \colon \mathcal{P}_{d,\mathbb{K}_N^{sp},E} \to \mathcal{A}_{n,d,N,E}$$

defined over E. We denote this morphism by KS. □

6.2 Extension in positive characteristic

In this section, we try to carry out the construction of the Kuga-Satake map in positive characteristic. While we cannot perform the construction in a direct way in positive characteristic, we can try to extend the Kuga-Satake map KS, originally defined over $\mathrm{Spec}(E)$, to an open part of $\mathrm{Spec}(\mathcal{O}_E)$. In order to do that, it is essential to refer to a result of Faltings.

Let R be a DVR of mixed characteristic $(0, p)$, with $p \nmid N$ and total index of ramification $e < p - 1$. Let η be the generic fiber and s be the special fiber. Let U be a smooth scheme over R.

Lemma 6.2.1. *Let $V \subset U$ be an open subscheme, containing the generic fiber U_η. Let $f_V \colon V \to \mathcal{A}_{n,d',N,R}$ be a morphism. Then f_V extends uniquely to a morphism $f'_V \colon V' \to \mathcal{A}_{n,d',N,R}$ for a maximal open subscheme of $V' \subset U$. Moreover, let x*

be a point in U and let L be the fraction field of $\mathcal{O}_{U,x}$. If the induced morphism $\mathrm{Spec}(L) \to \mathcal{A}_{n,d',N,R}$ *extends to a map*

$$\mathrm{Spec}(\mathcal{O}_{U,x}) \to \mathcal{A}_{n,d',N,R} \tag{6.2}$$

then the point x belong to V'.

Proof. The existence of a unique maximal open subscheme $V' \subset U$ such that the abelian scheme extends relies on the fact that abelian schemes are separated. The second part of the proposition is a technical argument, for which we refer to [Riz00, Lemma 4.3.3]. □

In particular, if the condition (6.2) is satisfied by the generic points of the special fiber U_s, we have that the complement $Z = U \setminus V'$ has codimension greater than 1. Thus, the following proposition concludes the argument for extending the morphism to all of U.

Lemma 6.2.2. *Let $V \subset U$ be an open subscheme, such that $U \setminus V$ has codimension greater than 1. Then, a morphism $f_V \colon V \to \mathcal{A}_{n,d',N,R}$ extends to $f_U \colon U \to \mathcal{A}_{n,d',N,R}$.*

Proof. The morphism $f_V \colon V \to \mathcal{A}_{n,d',N,R}$ corresponds to a triple $(a_V \colon A_V \to V, \mu_V, \beta_V)$ of an abelian scheme with polarization and level N structure. Under this assumption, as $U \setminus V$ has codimension greater than 1, we can apply the result of Faltings in [Moo98, Lemma 3.6], to extend the morphism a_V to all of U, obtaining an abelian scheme $a_U \colon A_U \to U$. The polarization $\mu_V \colon A_V \to A_V^\vee$ extends uniquely to a morphism $\mu_U \colon A_U \to A_U^\vee$, applying [FC90, Prop.2.7]. We need to check that μ_U defines a polarization. There exists an étale covering $\tilde{V} \to V$ such that the pullback $\mu_{\tilde{V}}$ of the polarization is defined by an ample line bundle $\mathcal{M}$ on $A_{\tilde{V}}$. By the Zariski-Nagata purity theorem, which applies to this case as $\mathrm{codim}(U \setminus V) \geq 2$, there exists an étale covering $\tilde{U} \to U$ such that $\tilde{V} = V \times_U \tilde{U}$. If we denote

$$\iota \colon \tilde{V} \to \tilde{U}, \qquad \iota_A \colon A_{\tilde{V}} \to A_{\tilde{U}},$$

the push-forward $\iota_{A*}\mathcal{M}$ is a line bundle on $A_{\tilde{U}}$, which is again ample, as it is ample for the fibers over $v \in \tilde{V}$, applying the result [Ray70, Cor. VIII.7]. The level structure β_V automatically extends to $a_U \colon A_U \to U$, as it is enough to define it on a geometric point of the basis. □

By this proposition, we can reduce the problem of extending the Kuga-Satake map KS in positive characteristic to the extension of the map on DVRs. Let r be the product of the primes p dividing $2dN$ or with ramification index greater or equal than $p-1$ in $\mathcal{O}_E$, where E is the number field over which we defined the Kuga-Satake map. The extension result is the following.

Theorem 6.2.3 (Kuga-Satake in positive characteristic)**.** *The Kuga-Satake map admits an extension*

$$\mathrm{KS} \colon \mathcal{P}_{d,\mathbb{K}_N^{sp},\mathcal{O}_E[1/r]} \to \mathcal{A}_{n,d,N,\mathcal{O}_E[1/r]}.$$

Proof. The restriction to the primes not dividing $2dN$ is necessary to apply the previous results, because we need to work with smooth schemes (compare with Rmk. 5.3.5). It suffices to define an extension for the DVR $R = \mathcal{O}_{E,\mathfrak{p}}$, for a prime $\mathfrak{p} \nmid r$. The separated algebraic space $\mathcal{P}_{d,\mathbb{K}_N^{sp},R}$ is smooth over R, so it has an étale covering

$$U \to \mathcal{P}_{d,\mathbb{K}_N^{sp},R},$$

where U is a smooth scheme over R. The Kuga-Satake morphism defined in 6.1.2 defines an abelian family with polarization and level structure $(a_{U_E} \colon A_{U_E} \to U_E, \mu_{U_E}, \beta_{U_E})$ over the generic fiber, given by the morphism $f_{U_E} \colon U_E \to \mathcal{A}_{g,d',N,R}$. Then, by Lemma 6.2.2, to show that the morphism extends to all of U amounts to proving that it extends to an open subscheme V such that $U \setminus V$ has codimension greater than or equal to 2. On the other hand, by Lemma 6.2.1, it suffices to show that we can extend the map to

$$\mathrm{Spec}(\mathcal{O}_{U,x}) \to \mathcal{A}_{n,d',N,R}$$

for all generic points x of the special fiber U_s. The local ring corresponding to the generic point of the special fiber $\mathcal{O}_{U,x}$ is regular and of dimension 1, therefore is a DVR. If we denote by L the fraction field of $\mathcal{O}_{U,x}$, L is an extension of the E. The morphism $\mathrm{Spec}(\mathcal{O}_{U,x}) \to U$ defines by pullback a triple

$$(\pi_{O_{U,x}} \colon X_{O_{U,x}} \to \mathrm{Spec}(O_{U,x}), \lambda_{O_{U,x}}, \alpha_{O_{U,x}})$$

of a K3 scheme with a polarization and a level structure. Taking again the pullback along the morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(\mathcal{O}_{U,x})$, we obtain a triple $(\pi_L \colon X_L \to \mathrm{Spec}(L), \lambda_L, \alpha_L)$, which defines a point $\mathrm{Spec}(L) \to \mathcal{P}_{d,\mathbb{K}_N^{sp},E}$. Then via the Kuga-Satake map, we obtain

$$\textsc{ks}(L)((\pi_L \colon X_L \to \mathrm{Spec}(L), \lambda_L, \alpha_L)) = (a_L \colon A_L \to \mathrm{Spec}(L), \mu_L, \beta_L).$$

Applying Thm. 4.3.5, we can see that the abelian variety $A_L \to \mathrm{Spec}(L)$ has potential good reduction over $\mathrm{Spec}(\mathcal{O}_{U,x})$. Arguing as in [And96, Lemma 9.3.1], we can see that the existence of a level structure β_L implies that the N-torsion points of A_L are L-rational, which, combined with the fact that A_L has potential good reduction, proves that A_L has good reduction, by [TS68, Cor. 2, page 497]. It remains to show that the polarization defined over $\mathrm{Spec}(L)$ extends over $\mathrm{Spec}(\mathcal{O}_{U,x})$. It is clear that $\mu_L \colon A_L \to A_L^\vee$ extends to a morphism $\mu_{\mathcal{O}_{U,x}} \colon A_{\mathcal{O}_{U,x}} \to A^\vee_{\mathcal{O}_{U,x}}$, which is in fact a polarization, applying Raynaud's result as in the proof of Lemma 6.2.2.

Finally, it remains to show that the morphism $U \to \mathcal{A}_{n,d',N,R}$ descends to a morphism $\mathcal{P}_{d,\mathbb{K}_N^{sp},\mathcal{R}} \to \mathcal{A}_{n,d,N,,R}$. This follows from the fact that the extension of a morphism $U_E \to \mathcal{A}_{n,d,N,,R}$ to all of U is unique, by Lemma 6.2.1. □

6.3 Conclusion

The result of Thm. 6.2.3 achieves the goal of defining the Kuga-Satake map as a morphism between the moduli spaces of K3 surfaces and the moduli space of abelian varieties and extend it in positive characteristic. In order to do so, we could not define a morphism over $\mathrm{Spec}(\mathbb{Z})$ directly; instead, we had to restrict ourselves to an open part of $\mathrm{Spec}(\mathcal{O}_E)$, where E is a number field.

Let us relate the Kuga-Satake map over moduli spaces to the constructions we discussed in Chapter 3. Following Deligne's approach, we saw that the Kuga-Satake construction works for families of complex K3 surfaces. In fact, considering complex families with an additional level structure, we need not pass to an étale covering to perform the construction. But, in order to define a Kuga-Satake morphism over moduli spaces, we would have to associate to a K3 surface over a number field an abelian variety over the field itself, while we could only prove that the Kuga-Satake variety descends to a finite extension of the field. This problem is not easy to overcome; for instance, André proves results in this direction in [And96, Thm 8.4.3], but they do not hold in full generality. Therefore, following Rizov's approach in [Riz00], we considered the question from a different perspective, and carried out the construction via Shimura varieties. The advantage of this point of view is that we can easily determine the field of definition of the map, using the theory of canonical models for Shimura varieties. In order to show that the Kuga-Satake map KS descends over a number field, the non-trivial point is proving the descent of the period map j. One way to achieve this is an argument of complex multiplication for K3 surfaces, also due to Rizov. This lets us determine explicitly the field of definition of the map

$$\mathrm{KS}\colon \mathcal{P}_{d,\mathbb{K}_N^{sp},\mathbb{C}} \to \mathcal{A}_{n,d',N,\mathbb{C}}$$

and extend it in positive characteristic for almost all primes.

One might wonder why the Kuga-Satake construction over moduli spaces was defined for polarized K3 surfaces and abelian varieties with a level structure. The reason is that, without a level structure, the Kuga-Satake map would not be well-defined even for complex K3 surfaces. Indeed, given a complex K3 surface we can attach to it an abelian variety, but we cannot determine a polarization uniquely. Even if we fix an element a in the Clifford algebra C satisfying $\iota(a) = -a$, we need to choose a marking of the Clifford algebra to define a polarization. Different choices of this marking give rise to non-isomorphic polarized abelian varieties. Considering, instead, K3 surfaces with a $\mathbb{K}_N^{\mathrm{sp}}$ -level structure lets us identify uniquely the polarization and level structure on the Kuga-Satake variety, because every choice of a marking of the Clifford algebra, up to multiplication by elements in $\mathbb{K}_N^{\mathrm{sp}}$, gives rise to the same polarization. This explains the fact that the Kuga-Satake construction works over moduli spaces with level structures.

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