Parabolic subgroups
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Overview

Tori

Fix k a field, G/k affine algebraic group (e.g. $G = \mathrm{GL}_{n,k}$, $\mathrm{Sp}_{n,k}$, $\mathrm{O}_{n,k} \ldots$)

Definition

A split torus T of G is subgroup of G isomorphic to $\mathbb{G}_m^n$ torus is $T \subset G$ such that $T_{\bar{k}}$ is a split torus.

- The diagonal subgroup $D_{n,k}$ is a maximal torus in $\mathrm{GL}_{n,k}$;
- The maximal torus in $\mathrm{Sp}_{n,k}$ is isomorphic to $\mathbb{G}_m^n$ under $(a_1, \ldots a_n) \mapsto \mathrm{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1})$;
- If F number field, $Res_{F/\mathbb{Q}}\mathbb{G}_m$ is a non-split torus .

For a torus T, we let

$$X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$$

be the space of characters of T. It's a finite free $\mathbb{Z}$-module.

Lie algebras and Adjoint representation

The Lie algebra $\mathfrak{g}$ of G is the tangent space of G at the identity: $\mathfrak{g} = T_eG$. It is an algebra with the product given by the Lie bracket $[X,Y] = XY - YX$ for $X,Y \in \mathfrak{g}$.
The action of G on itself by inner automorphism, induced an automorphism of the tangent space at the identity

$$\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$$

Assume $k = \bar{k}$, $T \subset G$ a maximal torus, G reductive. The vector space $\mathfrak{g}$ as a representation of T decomposes as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{t}$ is the Lie algebra of T.

Example: root system of SL_{n+1}

The Lie algebra $\mathfrak{sl}_{n+1}$ is given by

$$\{M \in M_{(n+1)\times(n+1)} \mid \mathrm{Tr}(M) = 0\}$$

Let $E_{i,j}$ be the matrix with 1 in the (i,j) entry and 0 elsewhere.
The maximal torus is $T \subset \mathrm{SL}_{n+1}$ given by $\mathbb{G}_m^n \subset \mathrm{SL}_{n+1}$

$$(t_1, t_2, \dots t_n) \mapsto (t_1, t_2, \dots t_n, (t_1 t_2 \dots t_n)^{-1})$$

For $1 \leq i, j \leq (n+1)$, $i \neq j$, define

$$\phi_{i,j} = a_{i,i} a_{j,j}^{-1}$$

where $a_{i,j}$ are the usual coordinate functions for $M_{(n+1)\times(n+1)}$.
Then $\forall t \in T$

$$\mathrm{Ad}(t) E_{i,j} = t E_{i,j} t^{-1} = \phi_{i,j}(t) E_{i,j}$$

$$\mathfrak{sl}_{n+1} = \mathfrak{t} \oplus \sum_{i,j} (\mathfrak{sl}_{n+1})_{\phi_{i,j}}$$

where $(\mathfrak{sl}_{n+1})_{\phi_{i,j}} = \langle E_{i,j} \rangle$.
The characters $\phi_{i,j} \in X(T)$ satisfy the relations:

1. $\phi_{i,j} = \phi_{j,i}^{-1}$
2. $\phi_{i,j} = \prod_{i \leq m < j} \phi_{m,m+1}$ if $i < j$.

Let $E = X(T) \otimes \mathbb{R} \simeq \mathbb{R}^n$. We see that

$$\alpha_i = \phi_{i,i+1}$$

is a basis of X; a_i can viewed as the vector $(0, \ldots, 0, 1, -1, \ldots)$ with 1 in the i-th position in $\mathbb{R}^{n+1}$. The $\phi_{i,j}$ span an n-dimensional subspace in $\mathbb{R}^{n+1}$.

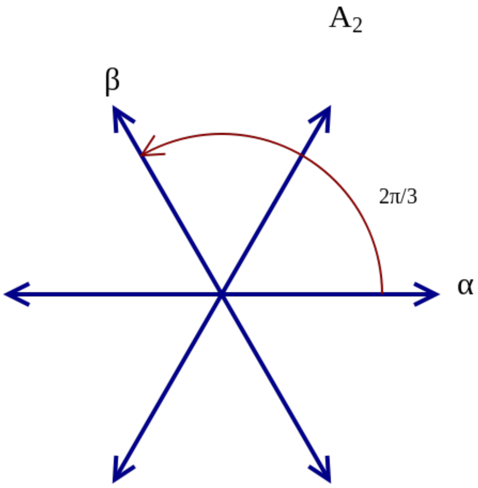
Denote by

$$\Delta = \{\alpha_i\} \subset \Phi = \{\phi_{i,j}\}$$

We have

$$\begin{aligned} \dim \mathrm{SL}_{n+1} &= (n+1)^2 - 1 = (n+1)n + n \\ &= |\Phi| + |\Delta| \\ &= |\Phi| + \mathrm{rk}(T) \end{aligned}$$

The root system of $\mathfrak{sl}_{n+1}$ is called A_n. For example, A_2 can be represented as in the picture.



Abstract root system

Let $E/\mathbb{R}$ be a finite vector space.

Definition

An (abstract) root system in the real vector space E is a subset Φ of E satisfying:

1. Φ is finite, spans E, and does not contain 0. (The elements of Φ are called roots.)
2. If $\alpha \in \Phi$, the only multiples of $\alpha \in \Phi$ are $\pm\alpha$.
3. If $\alpha \in \Phi$, there exists a reflection σ_α relative to α, which leaves Φ stable.
4. If then $\alpha, \beta \in \Phi$, $\sigma_\alpha(\beta) - \beta$ is an integral multiple of α.

The Weyl group of Ψ is

$$W = W(\Psi) = \langle \sigma_\alpha, \alpha \in \Psi \rangle \subset GL(E).$$

A subset $\Delta \subset \Psi$ is called a basis if $\Delta = \{\alpha_1, \ldots \alpha_n\}$ is a basis of E, relative to which each $\beta \in \Psi$ has a (unique) expression $\beta = \sum_i c_i \alpha_i$, where the c_i are integers of the same sign.
The Weyl group can be identified with $W = N_G(T)/T$, where $N_G(T)/T$ for a maximal split torus T in G.

Example

For example, for $G = \mathrm{SL}_{n+1}$, the Weyl group is S_n. The normalizer in G of the diagonal subgroup is the group of monomial matrices, i.e. matrices having exactly one non-zero element for each row and column.

Borel subgroups

Definition

A Borel $B \subset G$ subgroup is a maximal (closed) connected solvable subgroup.

For example, for $G = \mathrm{GL}_{n,k}$ a Borel subgroup is $T_{n,k}$, the subgroup of upper triangular matrices.

Theorem (Fixed point Theorem)

Let G be a connected solvable group acting on a complete variety X. Then G has a fixed point in X

Let V/k be a finite vector space, $\mathscr{F}(V)$ a flag complete flag variety. A connected solvable subgroup $G \subset \mathrm{GL}(V)$ acts on $\mathscr{F}(V)$ and the action has a fixed point. Thus, G stabilizes a complete flag in V!

Theorem

- *If $B \subset G$ is a Borel subgroup, then G/B is projective.*
- *All Borel subgroups are conjugate of each other, in particular of the same dimension.*

Example

If V/k is a vector space with basis $v_1, \dots v_n$, we fix the flag $F = (V_1 \subset V_2 \cdots \subset V_n)$.

$$V_i = \langle v_1, \dots v_i \rangle$$

Then F is a point in the flag variety $\mathscr{F}(V)$. If $G = \mathrm{GL}_n(k)$, then

$$\mathrm{Stab_G}(F) = T_{n,k} \qquad \mathrm{orbit}(F) = \mathscr{F}(V).$$

Thus, $G/B \simeq \mathscr{F}(V)$ is projective.

Definition of Parabolic subgroups

Definition

A closed subgroup $P \subset G$ is **parabolic** if the quotient G/P is projective.

Theorem

A closed subgroup P of G is parabolic if and only if it contains a Borel subgroup.

Proof.

"$\Rightarrow$" If P is parabolic, G/P is a complete variety, so B fixes a point in $G/P \Rightarrow gBg^{-1} \subset P$;
"$\Leftarrow$" If $B \subset P$, then $G/B \to G/P$ is a surjective map from a complete variety $\Rightarrow G/P$ is complete, hence projective (because G/P is always quasi-projective).

□

Tits systems

G group $\rightsquigarrow$ G reductive group
B, N subgroups of G such that G is generated by N, B $\rightsquigarrow$ B Borel, N normalizer of a maximal torus
$T = B \cap N$ normal in N $\rightsquigarrow$ T maximal torus
$W = N/T$ $\rightsquigarrow$ W Weyl group
$S \subset W$ a set of involutions generating W. $\rightsquigarrow$ $S = \Delta$

Definition

We say that (G, B, N, S) is a Tits system if

- If $\rho \in S$, $\sigma \in W$, $\rho B \sigma \subset B\sigma B \cup B\rho\sigma B$
- If $\rho \in S$, $\rho B \rho \neq B$.

Let (G, B, N, S) be a Tits system, $I \subset S$.

Theorem (Bruhat decomposition)

1. *If $\sigma, \sigma' \in W$, then $B\sigma B = B\sigma' B \iff \sigma = \sigma'$.*
2. $G = \sqcup_{\sigma \in W} B\sigma B$

In particular, For $I \subset S$, define $W_I = \langle \sigma \mid \sigma \in I \rangle$

$$P_I = BW_I B$$

- $I = \emptyset : P_\emptyset = B$ is the Borel subgroup
- $I = S$: $P_S = G$

Bruhat-Tits Theorem

Theorem

1. *The P_I's are parabolic subgroups of G*
2. *All the parabolic subgroups of G containing B are of the form P_I for some $I \subset S$.*
3. *If P_I and P_J are conjugate, then $I = J$.*

Corollary

Given a Borel subgroup $B \subset G$, there exist 2^r-parabolic subgroups containing B where r is the semisimple rank of G.

In particular, in order to determine all the parabolic subgroups, it suffices to find the maximal parabolic subgroups P_I of $I = S \setminus \{i\}$ and take their intersection.

Parabolic subgroups of $\mathrm{GL}_{n,k}$

The semisimple rank of GL_n is $n-1$. Fix a basis $v_1,\dots v_n$ and let $V_i = \langle v_1,\dots v_i\rangle$. Let F be the complete flag $F = (V_1 \subset V_2 \cdots \subset V_n)$. Then

Theorem

The parabolic subgroups of $\mathrm{GL}_{n,k}$ containing the standard Borel $B_{n,k}$ are the stabilizers of a subflag of F.

Let F_I with $I \subset \{1,\dots n-1\}$ be the flag obtained from V removing the subspaces of indices in I. Then P_I is the stabilizer of such flag in the flag variety of corresponding numerical invariants $\mathscr{F}(I)$. All the $P_I's$ look like ”upper triangular matrices with blocks on the diagonal”.

Parabolic subgroups of $\mathrm{Sp}_{n,k}$

Let V a k-vector space with a symplectic pairing Ψ. We can find a maximal isotropic basis $v_1, \dots v_n$ such that

$$V_i = \langle v_1, \dots v_i \rangle \subset \langle v_1, \dots v_i \rangle^{\perp}$$

so that the pairing is given by

$$\Psi = \begin{bmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{bmatrix}$$

A maximal torus in $T \subset \mathrm{Sp}$ is $\mathbb{G}_m^n \subset \mathrm{Sp}_{n,k}$ as

$$(t_1, t_2, \dots t_n) \mapsto \mathrm{diag}(t_1, t_2, \dots t_n, t_1^{-1}, t_2^{-1}, \dots t_n^{-1})$$

Thus, $\mathrm{rk}(G) = n$.

Theorem

The parabolic subgroups of $\mathrm{Sp}_{n,k}$ *are the stabilizers of isotropic flags.*

$I \subset \{1,...,n\}$, let $F_I = \{V_1, V_2, \ldots V_j \ldots\}$ be the subflag obtained by omitting V_i with $i \in I$ from the subflag.
G acts on the flag variety $\mathscr{F}(I)$ of numerical invariants $i \notin I$.
The orbit of F is the subvariety of isotropic subspaces and

$$P_I = \mathrm{Stab}_G(F_I).$$

All the P_I's are distinct, and determined by the numerical invariants of the flag.
These are all the parabolics, because the rank of G is n.

Parabolic subgroups of $U(n,m)$

Let $U(n,m)$ be the **real** algebraic group associated to a hermitian form on $\mathbb{C}^{n+m}$ with basis $v_1, v_2, \ldots v_{n+m}$

$$(u,v) = \bar{u}^t \Psi v \qquad \Psi = \begin{bmatrix} 0 & 0 & \mathbb{1}_m \\ 0 & \mathbb{1}_{n-m} & 0 \\ \mathbb{1}_m & 0 & 0 \end{bmatrix}$$

More precisely, $U(n,m)(\mathbb{R}) = \{M \in \mathrm{GL}_n(\mathbb{C}) \mid \bar{M}^t \Psi M = \Psi\}$. The maximal torus is isomorphic to $\mathbb{G}_m^m \times U(1)^{n-m}$. via $(t,u) \mapsto (t,u,t^{-1})$. The $\mathbb{R}$-semisimple rank of G is m.

Let $V_r = \langle e_1, \dots e_r \rangle \subset V_r^{\perp} = \langle e_1, e_2 \dots e_n, e_{n+r+1}, \dots e_{n+m} \rangle$ for $1 \leq r \leq m$.
Let $F = (V_1, \dots V_m)$ be the maximal isotropic flag.

Theorem

The parabolic subgroup P_I of $\mathrm{U}(n,m)$ is the stabilizers of isotropic subflags of F obtained by removing the V_i's for $i \in I$.

- The parabolic subgroups P_I are **real** subgroups, because the flags they stabilize is defined over $\mathbb{R}$. Since m is the rank of the maximal split torus over $\mathbb{R}$, these are all the real parabolic subgroups. In fact $U(n,m)$ can be defined over $\mathbb{Q}$ and the P_I's are **rational** subgroups.
- $U(n,m) \simeq U(n+m) \simeq \mathrm{GL}_{n+m,\mathbb{C}}$ has rank $n+m-1$, so has 2^{n+m-1} parabolic subgroups.

Thank you for your attention!